On the stable reduction of modular curves

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Abstract

We produce an integral model for the modular curve $X(Np^n)$ over the ring of integers of a sufficiently ramified extension of $\mathbf{Z}_p$ whose special fiber is a *semistable curve* in the sense that its only singularities are normal crossings. This is done by constructing a semistable covering (in the sense of Coleman) of the supersingular part of $X(Np^n)$ in a manner compatible with the transition maps. By "nonabelian Lubin-Tate theory", the cohomology of the tower $X(Np^n)$ realizes the local Langlands and Jacquet-Langlands correspondences for $\mathrm{GL}_2(\mathbf{Q}_p)$; we tie together nonabelian Lubin-Tate theory to the representation-theoretic point of view afforded by Bushnell-Kutzko types. Our work also applies to the Lubin-Tate tower of curves for a local field of positive characteristic, so that one obtains stable models for Drinfeld modular curves as well.

1. Introduction: The Lubin-Tate tower of curves

Let F be a non-archimedean local field with uniformizer π. The *Lubin-Tate tower* is a projective system $\mathfrak{X}(\pi^n)$ of rigid-analytic deformation spaces with π^n-level structure of a one-dimensional formal $\mathcal{O}_F$-module of height h over the residue field of F. (For precise definitions, see §3.1; for a comprehensive historical overview of Lubin-Tate spaces, see the introduction to [Str08a].) After extending scalars to a separable closure of F, the Lubin-Tate tower admits an action of the triple product group $\mathrm{GL}_h(F) \times B^\times \times W_F$, where B/F is the central division algebra of invariant $1/h$, and W_F is the Weil group of F. Significantly, the ℓ-adic étale cohomology of the Lubin-Tate tower realizes both the Jacquet-Langlands correspondence (between $\mathrm{GL}_h(F)$ and $B^\times$) and the local Langlands correspondence (between $\mathrm{GL}_h(F)$ and W_F). When $h = 1$, this statement reduces to classical Lubin-Tate theory [LT65]. For $h = 2$ the result was proved by Carayol ([Car83], [Car86]), who conjectured the general phenomenon under the name "non-abelian Lubin-Tate theory". Non-abelian Lubin-Tate theory was established for all h by Boyer [Boy99] for F of positive characteristic and by Harris-Taylor [HT01] for p-adic F. In both cases, the result is established by embedding F into a global field and appealing to results from the theory of Shimura varieties or Drinfeld modular varieties.

In this paper we focus on the case $h = 2$. In the case where $F = \mathbf{Q}_p$ for an odd prime p, or when F has odd positive characteristic, we construct a compatible family of *semistable models* $\mathscr{X}(\pi^n)$ for each $\mathfrak{X}(\pi^n)$ over the ring of integers of a sufficiently ramified extension L_n/F. For our purposes this means that $\mathscr{X}(\pi^n)$ is a formal scheme over $\mathcal{O}_{L_n}$ whose generic fiber is $\mathfrak{X}(\pi^n)$ and whose special fiber is a locally finitely presented scheme of relative dimension 1 over the residue field of L_n with only ordinary double points as singularities. This allows us to compute the cohomology of the Lubin-Tate tower of curves (along with the action of the three relevant

groups) by means of the weight spectral sequence; we therefore recover Carayol's result in a purely local manner.

The study of semistable models for modular curves begins with the Deligne-Rapoport model for $X_0(Np)$ in [DR73]. A semistable model for $X_0(Np^2)$ was constructed by Edixhoven in [Edi90]. A stable model for $X(p)$ was constructed by Bouw and Wewers in [BW04]. Most recently, a stable model for $X_0(Np^3)$ was constructed by Coleman and McMurdy in [CM10], using the notion of *semistable coverings* of a rigid-analytic curve by "basic wide opens". The special fiber of their model is a union of Igusa curves together which are linked at each supersingular point of $X_0(N) \otimes \mathbf{F}_p^{\mathrm{ac}}$ by a peculiar configuration of projective curves over $\mathbf{F}_p^{\mathrm{ac}}$, including in every case a number of copies of the curve with affine model $y^2 = x^p - x$. In each of these cases the interesting part of the special fiber of the modular curve is the supersingular locus. Inasmuch a Lubin-Tate curve (for $F = \mathbf{Q}_p$) appears as the rigid space attached to the p-adic completion of a modular curve at one of its mod p supersingular points, the problem of finding a semistable model for a modular curve is essentially the same as finding one for the corresponding Lubin-Tate curve. In this sense our result subsumes the foregoing results.

We now summarize our main result. Let F be either the field $\mathbf{Q}_p$ with p odd, or else the field of Laurent series $\mathbf{F}_q((\pi))$ with q odd. In either case, let π be a uniformizer of F, and let $k = \mathbf{F}_q$ be its residue field.

THEOREM 1.1. *For each $n \geqslant 1$, there is a finite extension $L_n/\hat{F}^{\mathrm{nr}}$ for which $\mathfrak{X}(\pi^n)$ admits a semistable model $\mathscr{X}(\pi^n)$; every connected component of the special fiber of $\mathscr{X}(\pi^n)$ admits a purely inseparable morphism to one of the following smooth projective curves over k^{ac}:*

(i) *The projective line $\mathbf{P}^1$,*
(ii) *The curve with affine model $xy^q - x^q y = 1$,*
(iii) *The curve with affine model $y^q + y = x^{q+1}$,*
(iv) *The curve with affine model $y^q - y = x^2$.*

REMARK 1.2. The mere existence of a semistable model of $\mathfrak{X}(\pi^n)$ (after passing to a finite extension of the field of scalars) follows from the corresponding theorem about proper (algebraic) curves. The rigid curve $\mathfrak{X}(\pi^n)$ is not proper, but it can be embedded into a proper curve (e.g., the appropriate global modular curve), and a semistable covering of the proper curve restricts to a semistable covering of $\mathfrak{X}(\pi^n)$. The content of the theorem is the assertion about the equations for the list of curves appearing therein. A semistable model is unique up to blowing up, so the above theorem holds for all semistable models of $\mathfrak{X}(\pi^n)$ if it holds for one of them.

REMARK 1.3. The equations for the curves appearing in Thm. 1.1 were known by S. Wewers to appear in the stable reduction of $\mathfrak{X}(\pi^n)$ (unpublished work).

The content of the paper goes far beyond the statement of Thm. 1.1. We devote the remainder of this introduction to explaining some other features of our semistable model $\mathscr{X}(\pi^n)$.

1.1 A semistable model for a tower of curves

Rather than focusing on any particular level along the Lubin-Tate tower, we construct a semistable model of the entire projective system of curves at once. This technique is discussed in §2.4 for a general tower of rigid curves admitting an action of a locally compact group G. In brief, a tower of curves X is a collection of rigid curves $X(K)$ indexed by the compact open subgroups $K \subset G$, together with quotient maps $X(K') \to X(K)$ for every inclusion $K' \subset K$. A semistable cover $\mathfrak{C}$

of X is a family of wide opens, each belonging to some level $X(K)$ of the tower, which satisfy a list of axioms. The wide opens are indexed by the vertices of a directed graph Γ; two vertices are adjacent if and only if the corresponding wide opens have preimages in some $X(K)$ which intersect. The axioms required are such that for each K one can use the wide opens to produce a semistable covering $\mathfrak{C}_K$ of $X(K)$. This semistable covering induces a formal model $\mathscr{X}(K)$ of $X(K)$.

The semistable coverings $\mathfrak{C}_K$ are compatible as one moves up the tower; if $K' \subset K$, the quotient map $X(K') \to X(K)$ extends to a finite morphism $\mathscr{X}(K') \to \mathscr{X}(K)$ of formal models. In particular one gets a finite morphism $\overline{\mathscr{X}}(K') \to \overline{\mathscr{X}}(K)$. One can ask about the projective limit of the reductions $\overline{\mathscr{X}}(K)$. Here one has no control in general over the behavior of irreducible components as one goes up the tower. Indeed one can imagine a scenario in which there is a descending chain K_n of open compact subgroups of G, and a chain $\cdots \to C_{n+1} \to C_n \to \ldots$ of irreducible components C_n of $\overline{\mathscr{X}}(K_n)$, for which the genus of C_n increases without bound. However, in the case of the Lubin-Tate tower of curves, something rather miraculous happens: no matter how the chain $\{C_n\}$ is chosen, the morphisms $C_{n+1} \to C_n$ are purely inseparable for n large enough. As a result the genus of C_n is bounded in any such chain, and the induced maps between étale cohomology groups $H^i(C_n, \mathbf{Q}_\ell)$ become isomorphisms for n large enough.

One arrives at a convenient combinatorial picture for the reduction of the Lubin-Tate tower in terms of a labeled graph. An "irreducible component" C of the reduction is a chain $\{C_n\}$ as above. The "dual graph" of the semistable reduction has one vertex for each connected component; there is an obvious notion of adjacency between vertices. The resulting dual graph has as uncountably many connected components; one connected component is displayed in §6.4.

In theory one could draw a picture of the stable reduction of any particular $\mathfrak{X}(\pi^n)$ by forming the quotient of the pictures described in §6.4 by the group $1 + \pi^n M_2(\mathcal{O}_F)$. However, doing this destroys the symmetry of the picture and adds unnecessary complexity even for $n = 2$.

1.2 The role of Bushnell-Kutzko types

The theory of types for $\mathrm{GL}_2(F)$ plays an indispensable role in our work. This theory, developed in broad generality by Bushnell-Kutzko in [BK93], furnishes an explicit parametrization of the supercuspidal representations of $\mathrm{GL}_h(F)$ by characters of finite-dimensional subgroups. There is a similar parametrization of the smooth irreducible representations of the multiplicative group of a central division algebra of dimension h^2 over F, see [Bro95]. In certain cases it is known how to align the two parametrizations according to the Jacquet-Langlands correspondence, see [Hen93], [BH05a], [BH05b].

The reduction of our semistable model of the Lubin-Tate tower is the union of closed subschemes C, each of which is a nonsingular (possibly disconnected) projective curve over the algebraic closure of the residue field of F. The stabilizer of any particular C in $\mathrm{GL}_2(F) \times B^\times$ is an open compact-mod-center subgroup of the form $E^\times \mathcal{L}^\times$, where $E \subset M_2(F) \times B$ is a quadratic extension field of F and $\mathcal{L} \subset M_2(F) \times B$ is a certain $\mathcal{O}_E$-order. The family of orders $\mathcal{L}$ was investigated in [Wei10], where they were called "linking orders"; we find that the representation of $E^\times \mathcal{L}^\times$ on $H^1(C, \mathbf{Q}_\ell)$ encodes the theory of types for $\mathrm{GL}_2(F)$ and $B^\times$ simultaneously, in the sense that the representation of $\mathrm{GL}_2(F) \times B^\times$ induced from $H^1(C, \mathbf{Q}_\ell)$ realizes the Jacquet-Langlands correspondence for those supercuspidal representations of $\mathrm{GL}_2(F)$ containing a certain class of simple strata corresponding to the choice of C. We therefore settle a question of Harris [Har02] on whether there exist analytic subspaces of the Lubin-Tate tower which realize the Bushnell-Kutzko types in their cohomology, at least in the case of GL_2.

1.3 Action of the Weil group

The reduction of our semistable model admits a k^{ac}-semilinear action of the Weil group W_F. A study of this action leads one to the notion of a "combinatorial local Langlands correspondence", whereby the components of the reduction, each nothing more than a smooth projective curve over k^{ac}, realize the local Langlands correspondence in their H^1. This would give a purely local proof of non-abelian Lubin-Tate theory in the case of GL_2. We do not specify the action of W_F on the reduction here; rather we provide the details in the preprint [Weia]. The higher genus curves C appearing in the statement of Thm. 1.1 all have the property that the eigenvalues of Frobenius on their H^1 (over whatever base) are all powers of q up to multiplication by a root of unity. The eigenvalues of Frobenius of these C are always Gauss sums.

For F of odd characteristic, one can give a complete classification of (1) irreducible Langlands parameters $W_F \to \mathrm{GL}_2(\mathbf{C})$ and (2) supercuspidal representations of $\mathrm{GL}_2(F)$ by means of *admissible pairs* $(E/F, \chi)$. However the translation between (1) and (2) is not the local Langlands correspondence; the discrepancy is measured by a character of $E^\times$ (see [BH05a], [BH05b]). We find that the discrepancy is manifested exactly in the eigenvalues of Frobenius on the curves C as above.

We strongly believe that the Lubin-Tate tower for GL_n contains analytic subspaces (specifically, affinoids with good reduction) which realize the local Langlands correspondence in their cohomology. For partial results in this direction, see [Weib].

1.4 Notational conventions

We reserve use of the overline ("$\overline{C}$") for the operation of passing to the special fiber. If F is a field, then F^{sep} and F^{ac} denote the separable and algebraic closures of F, respectively. If E is a nonarchimedean local field then $\mathfrak{p}_E$ is its maximal ideal and k_E is its residue field. The letter ℓ shall always denote a prime unequal to the characteristic of whatever geometric objects are nearby.

2. Stable reduction of a tower of curves: Generalities

2.1 Formal coverings and reductions

A rigid-analytic variety X admits no canonical reduction. Instead one has the notion of a reduction of X relative to a *formal covering*; this is a means of extending X to a formal scheme. We review these concepts from [BL85].

Let U be an affinoid variety. An admissible open affinoid subvariety $V \subset U$ is a *formal subdomain* if V is the inverse image of an open affine subset of $\overline{U}$ under the reduction map $U \to \overline{U}$. Now let X be a rigid-analytic variety. An admissible open affinoid covering $\mathfrak{U} = \{U_i\}_{i \in I}$ is a *formal covering* if, for each $i, j \in I$, the intersection $U_i \cap U_j$ is a finite union of formal subdomains of U_i.

Given a formal covering $\mathfrak{U} = \{U_i\}$ of X, one forms the reduction $\overline{X}_{\mathfrak{U}}$ as follows. Let $\pi_i \colon U_i \to \overline{U}_i$ be the reduction map. Then $\overline{X}_{\mathfrak{U}}$ is formed by gluing together the varieties $\overline{U}_i$ along the isomorphisms $\pi_i(U_i \cap U_j) \cong \pi_j(U_i \cap U_j)$.

2.2 Semistable reduction of curves

We review the notion of semistable model for algebraic curves.

DEFINITION 2.1. Let F be a nonarchimedean local field, and let X/F be a smooth proper curve.

A *semistable model* for X is a proper flat morphism $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_F$ whose generic fiber is X and whose geometric special fiber has at worst ordinary double points as singularities. A semistable model $\mathscr{X}$ is *stable* if in addition every rational component of the special fiber meets at least 3 other components.

REMARK 2.2. Note that we do not require that $\mathscr{X}$ be a regular scheme. Nor do we require that $\mathscr{X}$ be of finite type. For instance, if $X = \mathbf{P}_F^1$ then we can construct a semistable model $\mathscr{X}$ over $\mathcal{O}_F$ by performing infinitely many blow-ups on $\mathbf{P}_F^1$; then the special fiber of $\mathscr{X}$ will be an infinite union of rational components.

Now suppose F is discretely valued, with perfect residue field k. Write $S = \operatorname{Spec} \mathcal{O}_F$, $S^{\mathrm{sep}} = \operatorname{Spec} \mathcal{O}_{F^{\mathrm{sep}}}$, $\eta = \operatorname{Spec} F$, $\eta^{\mathrm{sep}} = \operatorname{Spec} F^{\mathrm{sep}}$, $s = \operatorname{Spec} k$, $s^{\mathrm{ac}} = \operatorname{Spec} k^{\mathrm{ac}}$. Suppose $\mathscr{X}$ is a semistable model of X; let $\mathscr{X}_{s^{\mathrm{ac}}} = \mathscr{X} \times_{\operatorname{Spec} \mathcal{O}_F} s^{\mathrm{ac}}$. The *dual graph* of $\mathscr{X}_{s^{\mathrm{ac}}}$ has a vertex for each irreducible component of $\mathscr{X}_{s^{\mathrm{ac}}}$ and an edge joining each pair of vertices whose corresponding components meet in $\mathscr{X}_{s^{\mathrm{ac}}}$.

The étale cohomology $H^1(X_{\eta^{\mathrm{sep}}}, \mathbf{Q}_\ell)$ is a finite-dimensional $\mathbf{Q}_\ell$-vector space admitting an action of $\operatorname{Gal}(F^{\mathrm{sep}}/F)$. The group $\operatorname{Gal}(F^{\mathrm{sep}}/F)$ also acts on $H^1(\mathscr{X}_{s^{\mathrm{ac}}}, \mathbf{Q}_\ell)$ and $H^1(\Gamma, \mathbf{Q}_\ell)$ through the quotient $\operatorname{Gal}(k^{\mathrm{ac}}/k)$. (Here $H_1(\Gamma, \mathbf{Q}_\ell)$ is the cohomology of the simplicial complex Γ.)

THEOREM 2.3. *There is an exact sequence of $\mathbf{Q}_\ell[\operatorname{Gal}(F^{\mathrm{sep}}/F)]$-modules*

$$0 \to H^1(\mathscr{X}_{s^{\mathrm{ac}}}, \mathbf{Q}_\ell) \to H^1(X_{\eta^{\mathrm{sep}}}, \mathbf{Q}_\ell) \to H^1(\Gamma, \mathbf{Q}_\ell)(-1) \to 0$$

Proof. We follow the argument in [Ill94], §3. There the model $\mathscr{X}$ is required to be a regular scheme, so that étale locally around each singular point of $\mathscr{X}_{s^{\mathrm{ac}}}$ it is isomorphic to $\mathcal{O}_F[x,y]/(xy - \pi)$. For the purposes of proving the theorem, we may replace $\mathscr{X}$ with a suitable blow-up which is regular. Indeed, blowing up will only alter $\mathscr{X}_{s^{\mathrm{ac}}}$ by adding new rational components; the graph Γ remains the same up to homotopy equivalence. Let Y_i, $i \in I$ be the irreducible components of $\mathscr{X}_{s^{\mathrm{ac}}}$.

Write Λ for the constant sheaf $\mathbf{Q}_\ell$ on $\mathscr{X}_{s^{\mathrm{ac}}}$. The complex of *vanishing cycles* on $\mathscr{X}_{s^{\mathrm{ac}}}$ is defined for $p \geqslant 0$ by

$$R^p \Psi(\Lambda) = \iota^* R^p j_*(\Lambda),$$

where $\iota \colon \mathscr{X}_{s^{\mathrm{ac}}} \hookrightarrow \mathscr{X}_{S^{\mathrm{ac}}}$ and $j \colon \mathscr{X}_{\eta^{\mathrm{sep}}} \hookrightarrow \mathscr{X}_{S^{\mathrm{sep}}}$ are the natural inclusions. The vanishing cycle complex is completely evaluated for semistable schemes $\mathscr{X}$ in [Ill94], Théorème 3.2(c). They are:

$$R^0 \Psi(\Lambda) = \Lambda$$

$$R^1 \Psi(\Lambda)(1) = \left(\bigoplus_{i \in I} \Lambda_{Y_i} \right) / \Lambda$$

$$R^p \Psi(\Lambda) = \bigwedge^p \Psi(\Lambda), \ p \geqslant 2$$

Here Λ_{Y_i} is the push-forward of the constant sheaf $\mathbf{Q}_\ell$ under $Y_i \hookrightarrow \mathscr{X}_{s^{\mathrm{ac}}}$.

In our case, where X is a curve, we have that $R^1 \Psi(\Lambda)$ is the skyscraper sheaf supported on the singular points of $\mathscr{X}_{s^{\mathrm{ac}}}$, with stalk $\mathbf{Q}_\ell$ at each such point. We have $R^p \Psi(\Lambda) = 0$ for $p \geqslant 2$.

Since $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_F$ is proper we have

$$H^p(\mathscr{X}_{s^{\mathrm{ac}}}, R^q \Psi(\Lambda)) = H^p(\mathscr{X}_{s^{\mathrm{ac}}}, \iota^* R^q j_*(\Lambda)) \cong H^p(X, R^q j_*(\Lambda)).$$

The Leray spectral sequence for j is therefore:

$$E_2^{p,q} = H^p(\mathscr{X}_{s^{\mathrm{ac}}}, R^q \Psi(\Lambda)) \implies H^{p+q}(X_{\eta^{\mathrm{ac}}}, \mathbf{Q}_\ell)$$

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The terms $E_2^{p,q}$ vanish for $\min(p,q) \geqslant 3$, so that the spectral sequence degenerates on page 3. We find that $H^1(X_{\eta^{\text{sep}}}, \mathbf{Q}_\ell)$ admits a filtration with quotients $E_3^{1,0} = E_2^{1,0}$ and $E_3^{0,1} = \ker(E_2^{0,1} \to E_2^{2,0})$. We have $E_2^{1,0} = H^1(\mathscr{X}_{s^{\text{ac}}}, \mathbf{Q}_\ell)$ by definition. We also have $E_2^{0,1} = H^0(\mathscr{X}_{s^{\text{ac}}}, R^1\Psi(\Lambda)) \cong C^1(\Gamma, \mathbf{Q}_\ell)(-1)$ and $E_2^{2,0} = H^2(\mathscr{X}_{s^{\text{ac}}}, \mathbf{Q}_\ell) \cong C^0(\Gamma, \mathbf{Q}_\ell)$, where $C^i(\Gamma, \mathbf{Q}_\ell)$ is the space of i-cochains in Γ with coefficients in $\mathbf{Q}_\ell$. The result follows. $\square$

2.3 Wide opens and semistable coverings

The following notions are taken from [Col03].

DEFINITION 2.4. A *wide open* (curve) is a rigid analytic space conformal to $C - D$, where C is a smooth complete curve and $D \subset C$ is a finite disjoint union of closed disks. Each connected component of C is required to contain at least one disk from D.

If W is a wide open, an *underlying affinoid* $Z \subset W$ is an affinoid subdomain for which $W \backslash Z$ is a finite disjoint union of annuli U_i. It is required that no annulus U_i be contained in any affinoid subdomain of W.

An *end* of W is an element of the inverse limit of the set of connected components of $W \backslash Z$, where Z ranges over affinoid subdomains of W.

Finally, W is *basic* if it has an underlying affinoid Z whose reduction $\overline{Z}$ is a semistable curve over $\overline{k}$.

We adapt the definition of semistable covering in [Col03], §2, which only applies to coverings of proper curves. Our intention is to construct semistable coverings of the spaces $\mathfrak{X}(\pi^n)$, which are wide open curves. Therefore we define:

DEFINITION 2.5. Let W be a wide open curve over a nonarchimedean field F. A *semistable covering* of W is an admissible covering $\mathcal{D}$ of W by connected wide opens satisfying the following axioms:

(i) If U, V are distinct wide opens in $\mathcal{D}$, then $U \cap V$ is a disjoint union of finitely many open annuli.

(ii) No three wide opens in $\mathcal{D}$ intersect simultaneously.

(iii) For each $U \in \mathcal{D}$, if

$$Z_U = U \backslash \left(\bigcup_{U \neq V \in \mathcal{D}} V \right),$$

then Z_U is a non-empty affinoid whose reduction is nonsingular.

In particular U is a basic wide open and Z_U is an underlying affinoid of U.

Suppose $\mathcal{D}$ is a semistable covering of a wide open W. For each $U \in \mathcal{D}$, let $A^0(U)$ be the ring of analytic functions on U of norm $\leqslant 1$, and let $X_U = \mathrm{Spf}\, A^0(U)$. Similarly if $U, V \in \mathcal{D}$ are overlapping wide opens, similarly define $X_{U \cap V} = \mathrm{Spf}\, A^0(U \cap V)$. Let $\mathscr{W}$ denote the formal scheme over $\mathcal{O}_F$ obtained by gluing the X_U together along the maps

$$X_{U,V} \to X_U \coprod X_V.$$

Then $\mathscr{W}$ has generic fiber W. The special fiber $\mathscr{W}_s$ of $\mathscr{W}$ is a scheme whose geometrically connected components are exactly the nonsingular projective curves $\overline{Z}_U^{\text{cl}}$ with affine model $\overline{Z}_U$; the curves $\overline{Z}_U^{\text{cl}}$ and $\overline{Z}_V^{\text{cl}}$ intersect exactly when U and V do.

EXAMPLE 2.6. Note that $\mathcal{D}$ will in general not be finite. Suppose W is the wide open disc $\operatorname{Sp} F[\![u]\!]$ over an algebraically closed field F; let $\pi \in F^{\times}$ be such that $0 < |\pi| < 1$. We construct a semistable covering $\mathcal{D} = \{U_n\}$ of W indexed by integers $n \geqslant 0$. First let $U_0 = \{|z| < |\pi^{1/2}|\}$, and for $n \geqslant 1$ let $U_n = \{|\pi|^{1/n} < |z| < |\pi|^{1/(n+2)}\}$. Then Z_{U_0} is the closed disk $\{|z| \leqslant |\pi|\}$ and for $n \geqslant 1$, Z_{U_n} is the "circle" $\{|z| = 1/(n+1)\}$. The resulting formal scheme $\mathcal{W}$ has special fiber which is an infinite union of rational components; the dual graph Γ is a ray.

Let C be the rigidification of a smooth complete curve, let $D \subset C$ be a disjoint union of closed discs, and let $W = C \backslash D$. A semistable covering of W yields a semistable covering of C (in the sense of [Col03]) by the following procedure. Let $\mathcal{D}$ be a semistable covering of W corresponding to the formal model $\mathcal{W}$. Let Γ be the dual graph attached to the special fiber of $\mathcal{W}$. There are bijections among the following three sets:

(i) ends of W,

(ii) ends of Γ, and

(iii) connected components of D.

Suppose $v_1, v_2, \ldots$ is a ray in Γ corresponding to the wide opens $U_1, U_2, \cdots \subset W$. Then there exists $N > 0$ such that for all $i \geqslant N$, U_i is an open annulus. If $D_0 \subset D$ is the connected component corresponding to the ray $v_1, v_2, \ldots$, then (possibly after enlarging N) $D_0 \cup \bigcup_{i \geqslant N} U_i$ is an open disc, which intersects U_{N-1} in an open annulus. Repeating this process for all ends of Γ yields a semistable covering of C by finitely many wide opens. Let $\mathcal{C} \to \operatorname{Spec} \mathcal{O}_F$ be the associated semistable model of C.

The following proposition shows how a semistable covering of a wide open curve W can be used to compute the compactly supported cohomology $H^1_c(W \otimes F^{\mathrm{sep}}, \mathbf{Q}_\ell)$.

PROPOSITION 2.7. *Let* C, D, W, $\mathcal{D}$, *and* $\mathcal{W}$ *be as above. Let* Y_i *($i \in I$) be the set of* non-rational *geometrically connected components of the special fiber of* $\mathcal{W}$, *so that each* Y_i *is a nonsingular projective curve of genus* $\geqslant 1$.

(i) *There is a diagram*

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H^1(\Gamma, \mathbf{Q}_\ell) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & H^1(\mathcal{C}_s, \mathbf{Q}_\ell) & \longrightarrow & H^1(C_\eta, \mathbf{Q}_\ell) & \longrightarrow & H^1(\Gamma, \mathbf{Q}_\ell)(-1) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \bigoplus_i H^1(Y_i, \mathbf{Q}_\ell) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

where the row and column are both exact.

(ii) *Let* $\operatorname{Ends}(\Gamma)$ *be the set of ends of the dual graph* Γ, *and let* $\mathcal{E}(\Gamma, \mathbf{Q}_\ell)$ *be the* $\mathbf{Q}_\ell$-*vector space*

with basis Ends(Γ). *There is an exact sequence*

$$0 \longrightarrow H^0(\Gamma, \mathbf{Q}_\ell) \xrightarrow{\text{diag.}} \mathcal{E}(\Gamma, \mathbf{Q}_\ell) \longrightarrow H^1_c(W \otimes F^{\text{sep}}, \mathbf{Q}_\ell) \longrightarrow H^1(C_{\eta^{\text{sep}}}, \mathbf{Q}_\ell) \longrightarrow 0$$

Proof. The horizontal sequence in the diagram in (1) is from Thm. 2.3. As for the vertical sequence: let $r \colon \tilde{\mathscr{C}}_s \to \mathscr{C}$ be the normalization, so that $\tilde{\mathscr{C}}_s$ is a disjoint union of nonsingular projective curves, containing the Y_i possibly along with some rational components. Consider the exact sequence of sheaves on $\mathscr{C}_s$:

$$0 \to \mathbf{Q}_\ell \to r_* r^* \mathbf{Q}_\ell \to \mathcal{Q} \to 0,$$

where the quotient $\mathcal{Q}$ is supported only on the singular points $x \in \mathscr{C}_s$, at which the stalk is one-dimensional. Part of the long exact sequence in cohomology reads

$$
\begin{array}{ccccccccc}
H^0(\mathscr{C}_s, r_* r^* \mathbf{Q}_\ell) & \longrightarrow & H^0(\mathscr{C}_s, \mathcal{Q}) & \longrightarrow & H^1(\mathscr{C}_s, \mathbf{Q}_\ell) & \longrightarrow & H^1(\mathscr{C}_s, r_* r^* \mathbf{Q}_\ell) & \longrightarrow & H^1(\mathscr{C}_s, \mathcal{Q}) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle =} \\
C^0(\Gamma, \mathbf{Q}_\ell) & \longrightarrow & C^1(\Gamma, \mathbf{Q}_\ell) & \longrightarrow & H^1(\mathscr{C}_s, \mathbf{Q}_\ell) & \longrightarrow & \bigoplus_{i \in I} H^1(Y, \mathbf{Q}_\ell) & \longrightarrow & 0,
\end{array}
$$

thus completing the diagram in part (1).

Part (2) follows directly from the long exact sequence in compactly supported cohomology for the pair (C, D). Note that $H^0_c(D, \mathbf{Q}_\ell) \cong \mathbf{Q}_\ell[\text{Ends}(\Gamma)]$. $\qquad\square$

2.4 Towers of curves

Let G be a locally profinite group (*e.g.*, $\mathrm{GL}_2(F)$).

DEFINITION 2.8. A *G-tower of rigid curves X* consists of the following data:

(i) A system $X(K)$ of rigid curves indexed by the compact open subgroups $K \subset G$, together with

(ii) Étale morphisms $\phi_g \colon X(K') \to X(K)$ whenever $K, K' \subset G$ are two subgroups with $g^{-1} K' g \subset K$; the ϕ_g are compatible in the obvious sense. When $K' \subset K$ is a normal subgroup, there is consequently an action of K/K' on $X(K')$. We require that $X(K') \to X(K)$ be the quotient of $X(K')$ by K/K'.

DEFINITION 2.9. A *dendritic filtration* of G is a G-equivariant directed cycle-free graph Γ on a vertex set V. To each vertex $v \in V$ there is an associated compact open subgroup $K_v \subset G$. The following properties must be satisfied:

(i) The association $v \mapsto K_v$ is compatible with the action of G in the sense that $K_{gv} = g K_v g^{-1}$ for each $g \in G$.

(ii) When $v, w \in V$ are adjacent vertices with $v \to w$, then $K_w \subset K_v$ is a proper normal subgroup.

(iii) Suppose $v_1 \to v_2 \to \cdots$ is an infinite directed path in Γ. Then $\bigcap_{n \geqslant 1} K_{v_n} = 1$.

For vertices $v, w \in V$, we write $v \preceq w$ if v and w can be joined in Γ by a directed path $v \to v_1 \to \cdots \to v_n \to v$ with $v_i \to v_{i+1}$ for $i = 1, \ldots, n-1$. Naturally, $v \prec w$ means $v \preceq w$ and $v \neq w$.

Whenever $v \preceq w$, write $\phi_{v,w}$ for the composition

$$X(K_w) \to X(K_{v_n}) \to \cdots \to X(K_{v_1}) \to X(K_v),$$

where $v \to v_1 \to \cdots \to v_n \to w$ is the (unique) path joining v to w in Γ.

We consider families of pairs (W_v, Z_v) indexed by $v \in V$, where $W_v \subset X(K_v)$ is a wide open and $Z_v \subset W_v$ is an underlying affinoid. For each v, let $U_v = W_v \backslash Z_v$, so that U_v is a disjoint union of annuli.

DEFINITION 2.10. Let X be a G-tower of wide open rigid curves, and let Γ be a dendritic filtration for G as in Defn. 2.9. A *semistable covering* of X with respect to Γ consists of the following data:

(i) For every vertex v of Γ, a wide open $W_v \subset X(K_v)$, and

(ii) For every vertex v and every edge e incident to v (in either direction), a disjoint union of open annuli $U_{v,e} \subset W_v$.

These data must satisfy the following requirements:

(SC0) (G-equivariance) For each $g \in G$, each vertex v, and each edge e incident to v, the pair $(W_{gv}, U_{gv,ge})$ is the image of $(W_v, U_{v,e})$ under the isomorphism $X(K_v) \to X(K_{gv})$.

(SC1) (Compatibility under edge maps) For each edge $e \colon v \to w$, we have

$$\phi_{v,w}(U_{w,e}) = U_{v,e}.$$

(SC2) (Each W_v is a basic wide open.) For every vertex v of Γ, let $Z_v \subset W_v$ be obtained by removing the annuli $U_{v,e}$ for each edge e incident to v:

$$Z_v = W_v \backslash \bigcap_e U_{v,e}.$$

Then $Z_v \subset W_v$ is an underlying affinoid, and the reduction $\overline{Z}_v$ is geometrically connected and nonsingular. In particular W_v is a basic wide open.

(SC3) (Persistence of basic wide opens under inverse images) For every pair of vertices v, w with $v \prec w$, and every edge e incident to v, the inverse image $\phi_{v,w}^{-1}(U_{v,e}) \subset X(K_v)$ is a disjoint union of open annuli. Furthermore, the inverse image $\phi_{v,w}^{-1}(Z_v)$ is an affinoid with good reduction. Therefore $\phi_{v,w}^{-1}(W_v)$ is a basic wide open with underlying affinoid $\phi_{v,w}^{-1}(Z_v)$.

(SC4) (The W_v really do cover X, with the appropriate intersections) For any point $y \in X$, exactly one of the two following conditions holds:

(a) There exists a unique vertex v such that the image of y in $X(K_v)$ lies in Z_v.

(b) There exists a unique edge $e \colon v \to w$ such that the image of y in $X(K_v)$ lies in $U_{v,e}$.

If a semistable covering of the G-tower X is given as above, then we may construct a semistable covering of $X(K)$ (in the sense of §2.5) for any $K \subset G$ compact open; the wide opens which cover $X(K)$ will be indexed by vertices in the quotient graph $K \backslash G$.

The construction runs as follows. For each vertex v, choose a vertex $w \succeq v$ for which $K_w \subset K$. (Such a vertex w exists by condition (3) in Defn. 2.9.) By condition (SC3), the preimage $\phi_{v,w}^{-1}(W_v)$ is a basic wide open in $X(K_w)$. Let W_{Kv} be the image of $\phi_{v,w}^{-1}(W_v)$ under the map $X(K_w) \to X(K)$; then $W_{Kv} \subset X(K)$ is a wide open which only depends on the coset Kv in $K \backslash \Gamma$. (In particular it does not depend on the choice of vertex w.)

PROPOSITION 2.11. *The wide opens W_{Kv}, where Kv runs through vertices of the quotient graph $K \backslash \Gamma$, constitute a semistable covering of $X(K)$.*

Proof. Our first claim is that for distinct vertices Kv, Kw of $K \backslash \Gamma$, W_{Kv} and W_{Kw} intersect if and only if Kv and Kw are adjacent in $K \backslash \Gamma$. But this follows directly from condition (SC4).

Since $K\backslash\Gamma$ has no cycles, no three distinct vertices could be pairwise adjacent, and this shows that no three distinct wide opens in our covering have nonempty intersection.

Suppose Kv and Kw are joined by an edge Ke of $K\backslash\Gamma$. After renaming v and w we may assume $e\colon v\to w$ is an edge of Γ. Let $w'\succeq w$ be large enough so that $K_{w'}\subset K_w$; then by our definitions, $W_{Kv}\cap W_{Kw}$ is the image of $\phi_{v,w'}^{-1}(U_{v,e})$ under $X(K_{w'})\to X(K)$. By condition (SC3), $\phi_{v,w'}^{-1}(U_{v,e})$ is an open annulus. Therefore $W_{Kv}\cap W_{Kw}$ is also an open annulus, c.f. Lemma 1.4 of [Col03].

Finally, for each vertex Kv in $K\backslash\Gamma$, consider the complement

$$Z_{Kv}=W_{Kv}\backslash\bigcup_{Kw\neq K_v}W_{Kw},$$

where the intersection runs over all vertices of $K\backslash\Gamma$ other than K_v. Let $w\succeq v$ be large enough so that $K_w\subset K$; then by (SC4), Z_{Kv} is exactly the image of $\phi_{v,w}^{-1}(Z_v)$ under $X(K_w)\to X(K)$. By (SC2) or (SC3) (as $w=v$ or not, respectively), $\phi_{v,w}^{-1}(Z_v)$ has good reduction. By Prop. 1.5 of [Col03], Z_{Kv} has good reduction as well. $\qquad\square$

The reduction of $X(K)$ with respect to $\mathfrak{U}$ will be denoted $\overline{X}(K)_{\mathfrak{U}}$.

DEFINITION 2.12. The semistable covering $\mathfrak{U}$ is *coherent* if the following condition holds: For every pair of vertices $v\prec w$, and every connected component $\overline{Z}_{v,w}$ of $\phi_{v,w}^{-1}(Z_v)$, the morphism $\overline{Z}_{v,w}\to\overline{Z}_v$ is purely inseparable.

For each vertex v in Γ, let $v=v_0\to v_1\to\dots$ be a ray in Γ, and for each $n\geqslant 1$ let $Z_{v,n}$ be a any connected component of $\phi_{v,v_n}^{-1}(Z_v)$; do this in such a way that the image of $Z_{v,n+1}$ in $X(K_{v_n})$ is $Z_{v,n}$. Then each map $\overline{Z}_{v,n+1}\to\overline{Z}_{v,n}$ is purely inseparable. Let $H_v\subset K_v$ be the stabilizer in G of the tower $\dots\to Z_{v,1}\to Z_{v,0}=Z_v$.

Also let

$$\mathcal{E}(\Gamma,\mathbf{Q}_\ell)=\varinjlim\mathcal{E}(K\backslash\Gamma,\mathbf{Q}_\ell),$$

where the limit is taken over compact open subgroups $K\subset G$. That is, $\mathcal{E}(\Gamma,\mathbf{Q}_\ell)$ is the space of $\mathbf{Q}_\ell$-valued functions on the (possibly infinite) set $\mathrm{Ends}(\Gamma)$ which are "smooth"; *i.e.* which are K-invariant for some compact open $K\subset G$.

The semistable coverings of $X(K)$ provided by Prop. 2.11 have contractible dual graphs. By Prop. 2.7 we can compute the cohomology of the tower X in terms of the components $\overline{Z}_v$:

PROPOSITION 2.13. *There is an exact sequence of G-modules*

$$0\to H^0(\Gamma,\mathbf{Q}_\ell)\to\mathcal{E}(\Gamma,\mathbf{Q}_\ell)\to\varinjlim H_c^1(X(K),\mathbf{Q}_\ell)\to\bigoplus_v\mathrm{Ind}_{H_v}^G\mathrm{Res}_{H_v}^{K_v}H^1(\overline{Z}_v^{\mathrm{cl}},\mathbf{Q}_\ell)\to 0$$

3. The Lubin-Tate tower of curves

3.1 Moduli of one-dimensional formal modules

Let F be a nonarchimedean field with ring of integers $\mathcal{O}_F$, uniformizer π, and residue field $k=\mathbf{F}_q$. Let Σ be a one-dimensional formal $\mathcal{O}_F$-module over k^{ac} of height 2. Let $\mathcal{C}$ be the category of complete local Noetherian $\hat{\mathcal{O}}_F^{\mathrm{nr}}$-algebras with residue field k^{ac}. We consider the moduli problem $\mathscr{M}^{(0)}$ which associates to each $R\in\mathcal{C}$ the set of isomorphism classes of *deformations* of Σ. A deformation of Σ to R is a pair $(\mathscr{F},\iota)$, where $\mathscr{F}$ is a formal $\mathcal{O}_F$-module over R and $\iota\colon\Sigma\tilde{\to}\mathscr{F}\otimes_R k^{\mathrm{ac}}$

is an isomorphism. An isomorphism between pairs $(\mathscr{F}, \iota)$ and $(\mathscr{F}', \iota')$ is an isomorphism of formal $\mathcal{O}_F$-modules $f: \mathscr{F} \to \mathscr{F}'$ which interlaces ι with ι'.

By [Dri74], Prop. 4.2, the functor $\mathscr{M}^{(0)}$ is represented by the formal scheme $\operatorname{Spf} \mathcal{A}$, where $\mathcal{A}$ is (noncanonically) isomorphic to the power series ring $\hat{\mathcal{O}}_F^{\mathrm{nr}}[\![u]\!]$ in one variable. Let $\mathscr{F}^{\mathrm{univ}}$ be the universal one-dimensional formal module over $\mathcal{A}$. Then if $A \in \mathcal{C}$ has maximal ideal M, then isomorphism classes of deformations of Σ to A are given exactly by specializing u to an element of M in $\mathscr{F}^{\mathrm{univ}}$.

3.2 Heights and quasi-isogenies

Let $R \in \mathcal{C}$. An isogeny $\iota: \mathscr{F} \to \mathscr{F}'$ between formal $\mathcal{O}_F$-modules over R has F-height $\operatorname{height}_F(\iota) = h$ if $\ker \iota$ is a group scheme over R of rank q^h. If $\alpha \in \mathcal{O}_F$ is nonzero, then the isogeny $[\alpha]: \mathscr{F} \to \mathscr{F}$ has height $2v_F(\alpha)$, where v_F is the usual valuation on F.

It will be convenient to expand the moduli problem of §3.1 to include pairs $(\mathscr{F}, \iota)$, where ι is not necessarily an isogeny but only a *quasi-isogeny*, which means that

$$\iota \in \operatorname{Hom}_{k^{\mathrm{ac}}} \left(\Sigma, \mathscr{F} \otimes k^{\mathrm{ac}} \right) \otimes_{\mathcal{O}_F} F$$

and that $\pi^r \iota$ is a (true) isogeny $\Sigma \to \mathscr{F} \otimes k^{\mathrm{ac}}$ for some $r \in \mathbf{Z}$. The F-height of ι is then defined to be $\operatorname{height}_F(\pi^r \iota) - 2r$.

Let $\mathscr{M}$ be the moduli problem which classifies pairs $(\mathscr{F}, \iota)$, where ι is a quasi-isogeny. For $h \in \mathbf{Z}$ let $\mathscr{M}^{(h)}$ be the sub-problem which classifies pairs $(\mathscr{F}, \iota)$ for which the quasi-isogeny ι has height h. Then

$$\mathscr{M} = \coprod_{h \in \mathbf{Z}} \mathscr{M}^{(h)}$$

3.3 Action of the division algebra

Let $\mathcal{O}_B = \operatorname{End}_{k^{\mathrm{ac}}} \Sigma$. Then $B = \mathcal{O}_B \otimes_{\mathcal{O}_F} F$ is the division algebra over F with invariant $1/2$, and $\mathcal{O}_B$ is its ring of integers. There is a right action of $B^\times$ on $\mathscr{M}$ given by $(\mathscr{F}, \iota)^b = (\mathscr{F}, \iota \circ b)$ for $b \in B^\times$. Let $N_{B/F}: B \to F$ be the reduced norm. Since $b: \Sigma \to \Sigma$ has F-height $v_F(N_{B/F}(b))$, we see that the action of b maps $\mathscr{M}^{(h)}$ isomorphically onto $\mathscr{M}^{h+v_F(N_{B/F}(b))}$.

3.4 Level structures

For an algebra $A \in \mathcal{C}$ with maximal ideal M and $(\mathscr{F}, \iota) \in \mathscr{M}(A)$, a *Drinfeld level π^n structure* on $\mathscr{F}$ is an $\mathcal{O}_F$-module homomorphism

$$\phi: (\pi^{-n} \mathcal{O}_F / \mathcal{O}_F)^{\oplus 2} \to M$$

for which the relation

$$\prod_{x \in (\mathfrak{p}^{-1}/\mathcal{O}_F)^{\oplus 2}} (X - \phi(x)) \,\Big|\, [\pi]_{\mathscr{F}}(X)$$

holds in $A[\![X]\!]$. The images under ϕ of the standard basis elements $(\pi^{-n}, 0)$ and $(0, \pi^{-n})$ of $(\pi^{-n}/\mathcal{O}_F)^{\oplus 2}$ form a *Drinfeld basis* of $\mathscr{F}[\pi^n]$.

Let $\mathscr{M}_n$ denote the functor which assigns to each $A \in \mathcal{C}$ the set of deformations $(\mathscr{F}, \iota)$ of Σ to A together with a Drinfeld level π^n structure on $\mathscr{F}$ over A. Of course $\mathscr{M}_n = \coprod_{h \in \mathbf{Z}} \mathscr{M}_n^{(h)}$, where $\mathscr{M}_n^{(h)}$ classifies triples $(\mathscr{F}, \iota, \phi)$ for which $\operatorname{height}_F(\iota) = h$.

By [Dri74], Prop. 4.3, the functor $\mathscr{M}_n^{(0)}$ is represented by a formal scheme $\operatorname{Spf} \mathcal{A}(\pi^n)$, where

$\mathcal{A}(\pi^n)$ is finite, flat, and generically étale over $\mathcal{A} \cong \hat{\mathcal{O}}_F^{\mathrm{nr}}[\![u]\!]$.

3.5 The limit problem, action of $\mathrm{GL}_2(F)$

The moduli problem $\mathscr{M}_n$ has a right actin of $\mathrm{GL}_2(\mathcal{O}_F/\pi^n\mathcal{O}_F)$ given by $(\mathscr{F}, \iota, \phi)^g = (\mathscr{F}, \iota, \phi \circ g)$. These actions coalesce into an "action" of $G = \mathrm{GL}_2(F)$ on the projective system

$$\mathscr{M} = \varprojlim \mathscr{M}_n.$$

Let $\mathfrak{X}$ be the system of rigid curves attached to the generic fiber of $\mathscr{M}$. Let $\mathbf{C}_F$ be the completion of F^{sep}: then a $\mathbf{C}_F$-point of $\mathfrak{X}$ corresponds to a triple $(\mathscr{F}, \iota, \phi)$, where $\mathscr{F}$ is a formal $\mathcal{O}_F$-module, $\iota \colon \mathscr{F}_0 \to \mathscr{F} \otimes k^{\mathrm{ac}}$ is a quasi-isogeny, and $\phi \colon F^{\oplus 2} \to V(\mathscr{F})$ is a trivialization of the rational Tate module of $\mathscr{F}$. From this point of view one has obvious actions of $\mathrm{GL}_2(F)$ and $B^\times$ on $\mathfrak{X}$.

DEFINITION 3.1. The *Lubin-Tate* tower of curves is the projective system $\{\mathfrak{X}(\pi^n)\}_{n \geqslant 0}$, where $\mathfrak{X}(\pi^n)$ is the rigid analytic curve attached to the generic fiber of $\mathscr{M}_n^{(0)}$. For an open subgroup $K \subset \mathrm{GL}_2(\mathcal{O}_F)$, let $\mathfrak{X}(K) = \mathfrak{X}(\pi^n)/K$ for n large enough so that $1 + \pi^n M_2(\mathcal{O}_F) \subset K$.

Each $\mathfrak{X}(\pi^n)$ is a wide open curve over $\hat{F}^{\mathrm{nr}}$.

3.6 Connected components of $\mathfrak{X}(\pi^n)$

Let LT be a one-dimensional formal $\mathcal{O}_F$-module over $\hat{\mathcal{O}}_{F^{\mathrm{nr}}}$ for which $\mathrm{LT} \otimes k^{\mathrm{ac}}$ has height one; this is unique up to isomorphism. Let $F_0 = \hat{F}^{\mathrm{nr}}$, and for $n \geqslant 1$, let $F_n = F_0(\mathrm{LT}[\pi^n])$. Then by classical Lubin-Tate theory, F_n/F_0 is an abelian extension with group $(\mathcal{O}_F/\pi^n\mathcal{O}_F)^\times$, and $\bigcup_{n \geqslant 1} F_n$ is the completion of the maximal abelian extension of F.

The geometrically connected components of $\mathfrak{X}(\pi^n)$ are defined over F_n, and they are in bijection with primitive elements in the free rank 1 $(\mathcal{O}_F/\pi^n\mathcal{O}_F)$-module $\mathrm{LT}[\pi^n]$. For the following theorem, we refer to Strauch [Str08b]:

THEOREM 3.2. *For each $n \geqslant 1$, there exists a locally constant rigid analytic function*

$$\Delta^{(n)} \colon \mathfrak{X}(\pi^n) \to \mathrm{LT}[\pi^n]$$

which surjects onto the subset of $\mathrm{LT}[\pi^n]$ which are primitive (i.e. not divisible by π). For a triple $(g, b, \tau) \in \mathrm{GL}_2(\mathcal{O}_F) \times \mathcal{O}_B^\times \times \mathrm{Gal}(\hat{F}^{\mathrm{ac}}/F_0)$, we have

$$\Delta^{(n)} \circ (g, b, \tau) = [\delta(g, b, \tau)]_{\mathrm{LT}} \left(\Delta^{(n)} \right),$$

where δ is the homomorphism

$$\delta \colon \mathrm{GL}_2(F) \times B^\times \times W_F \to F^\times$$
$$(g, b, w) \mapsto \det g \times N_{B/F}(b)^{-1} \times \left(\mathrm{Art}_F^{-1} w \right)^{-1}$$

and $\mathrm{Art}_F \colon F^\times \to W_F^{\mathrm{ab}}$ is the reciprocity map from local class field theory, normalized so that $\mathrm{Art}_F(\pi)$ is a geometric Frobenius element. The geometric fibers of $\Delta^{(n)}$ are connected.

3.7 Non-abelian Lubin-Tate theory

Let $\pi \mapsto \mathrm{LLC}(\pi)$ be the bijection between irreducible admissible representations of $\mathrm{GL}_2(F)$ (with $\mathbf{C}$ coefficients) and two-dimensional Frobenius-semisimple Weil-Deligne representations of F afforded by the local Langlands correspondence. Write $\mathcal{H}(\pi) = \mathrm{LLC}(\pi \otimes |\det|^{-1/2})$; then

$\pi \mapsto \mathcal{H}(\pi)$ is compatible under automorphisms of $\mathbf{C}$; it may therefore be extended unambiguously to representations with coefficients in any algebraically closed field of characteristic zero.

Let
$$H_c^1 = H_c^1 \left(\mathfrak{X} \otimes_{\hat{F}^{\mathrm{nr}}} \mathbf{C}_F, \mathbf{Q}_\ell^{\mathrm{ac}} \right),$$
so that H_c^1 admits an action of $\mathrm{GL}_2(F) \times B^\times \times W_F$.

THEOREM 3.3. *Let π be an irreducible essentially square-integrable representation of $\mathrm{GL}_2(F)$, with coefficients in $\mathbf{Q}_\ell^{\mathrm{ac}}$.*

(i) *If π is supercuspidal, then*
$$\mathrm{Hom}_{\mathrm{GL}_2(F)}(H_c^1, \pi) \cong \mathrm{JL}(\pi) \otimes \mathcal{H}(\tilde{\pi}).$$

(ii) *If $\pi \cong \mathrm{St} \otimes (\chi \circ \det)$, where St is the Steinberg representation and χ is a character of $F^\times$, then*
$$\mathrm{Hom}_{\mathrm{GL}_2(F)}(H_c^1, \pi) \cong \mathrm{JL}(\pi) \otimes (\chi^{-1} \circ \mathrm{Art}_F^{-1}).$$

3.8 Convenient models for formal modules

Recall that Σ is our fixed formal $\mathcal{O}_F$-module of height 2 over k^{ac}. If $\mathrm{char}\, F = 0$ it seems difficult to give explicit power series for the structure of Σ, let alone for the structure of the universal deformation $\mathscr{F}^{\mathrm{univ}}$ over $\hat{\mathcal{O}}_F^{\mathrm{nr}}[\![u]\!]$. However, suppose for the moment that $\mathrm{char}\, F = p > 0$, so that $\mathcal{O}_F = k[\![\pi]\!]$, where k is the field of q elements. A general rubric for producing a formal $\mathcal{O}_F$-module $\mathscr{F}$ (over an $\mathcal{O}_F$-algebra R, say) goes as follows:

$$X +_{\mathscr{F}} Y = X + Y$$
$$[\zeta]_{\mathscr{F}}(X) = \zeta X, \ \zeta \in k$$
$$[\pi]_{\mathscr{F}}(X) = \text{any } k\text{-linear power series } i(\pi)X + a_1 X^q + a_2 X^{q^2} + \dots.$$

Here $i \colon \mathcal{O}_F \to R$ is the structure homomorphism.

In particular we find a model for the unique one-dimensional formal $\mathcal{O}_F$-module Σ over k^{ac} of height 2:
$$[\pi]_\Sigma(X) = X^{q^2}$$
We can also model the universal deformation of Σ over $\mathcal{A} = \hat{\mathcal{O}}_F^{\mathrm{nr}}[\![u]\!]$:
$$[\pi]_{\mathscr{F}^{\mathrm{univ}}}(X) = \pi X + u X^q + X^{q^2},$$
see [Str08a], Prop. 5.1.1 (specialization to the case $n = 2$).

If $\mathrm{char}\, F = 0$, Hazewinkel's theory of "typical formal modules" ([Haz78]) ensures that it is possible to choose a coordinate X on $\mathscr{F}^{\mathrm{univ}}$ for which the following congruences hold in $\mathcal{A}[\![X]\!]$:

$$[\pi]_{\mathscr{F}^{\mathrm{univ}}}(X) = u X^q \quad (\mathrm{mod}\ \pi, \deg q + 1)$$
$$[\pi]_{\mathscr{F}^{\mathrm{univ}}}(X) = X^{q^2} \quad (\mathrm{mod}\ \pi, u, \deg q^2 + 1).$$

3.9 The boundary of the Lubin-Tate tower

As one approaches the boundary $\{|u| = 1\}$ of the disc $\mathfrak{X}(1)$, the corresponding formal $\mathcal{O}_F$-module scheme $\mathscr{F}[\pi^n]$ admits a canonical subgroup. If a Drinfeld level structure on $\mathscr{F}[\pi^n]$ is given, this canonical subgroup can be pulled back to a line in $(\mathcal{O}_F/\pi^n \mathcal{O}_F)^{\oplus 2}$, which is to say an element of $\mathbf{P}^1(\mathcal{O}_F/\pi^n \mathcal{O}_F)$. One can prove:

Proposition 3.4. *The ends of $\mathfrak{X}(\pi^n)$ are in bijection with $\mathbf{P}^1(\mathcal{O}_F/\pi^n\mathcal{O}_F)$. Any semistable model for $\mathfrak{X}(\pi^n)$ has contractible dual graph.*

3.10 Some coordinates on the Lubin-Tate tower

For each $n \geqslant 1$, let $X_1^{(n)}, X_2^{(n)} \in \mathcal{A}(\pi^n)$ be the Drinfeld basis for $\mathscr{F}^{\mathrm{univ}}[\pi^n]$ corresponding to the tautological Drinfeld level π^n level structure on $\mathscr{F}^{\mathrm{univ}}[\pi^n]$ over $\mathscr{M}_n$. Then $\left\{X_1^{(n)}, X_2^{(n)}\right\}$ is a set of regular parameters for $\mathcal{A}(\pi^n)$ [Dri74].

The group $\mathrm{GL}_2(\mathcal{O}_F)$ acts on $\mathfrak{X}(\pi^n)$ on the left, so it acts on the right on $\mathcal{A}(\pi^n)$. We write this action simply as $M(X)$ for $M \in \mathrm{GL}_2(\mathcal{O}_F)$, $X \in \mathcal{A}(\pi^n)$. In terms of the coordinates $X_i^{(n)}$, the action works the following way:

$$M\left(X_1^{(n)}\right) = [a]_{\mathscr{F}^{\mathrm{univ}}}\left(X_1^{(n)}\right) +_{\mathscr{F}^{\mathrm{univ}}} [c]_{\mathscr{F}^{\mathrm{univ}}}\left(X_2^{(n)}\right)$$
$$M\left(X_2^{(n)}\right) = [b]_{\mathscr{F}^{\mathrm{univ}}}\left(X_1^{(n)}\right) +_{\mathscr{F}^{\mathrm{univ}}} [d]_{\mathscr{F}^{\mathrm{univ}}}\left(X_2^{(n)}\right)$$

It will be tremendously useful to extend the definition of $M\left(X_i^{(n)}\right)$ to include arbitrary matrices $M \in M_2(F)$. If $\pi^m M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_F)$, we define

$$M\left(X_1^{(n)}\right) = [a]_{\mathscr{F}^{\mathrm{univ}}}\left(X_1^{(m+n)}\right) +_{\mathscr{F}^{\mathrm{univ}}} [c]_{\mathscr{F}^{\mathrm{univ}}}\left(X_2^{(m+n)}\right)$$

and similarly for $X_2^{(n)}$. Of course the definition is independent of m.

3.11 Canonical lifts

Here we discuss the notion of canonical lift introduced by Gross in [Gro86].

Definition 3.5. *A point x of $\mathfrak{X}$ represented by a triple $(\mathscr{F}, \iota, \phi)$ is a canonical lift if $E = \mathrm{End}\,\mathscr{F} \otimes_{\mathcal{O}_F} F$ is a separable quadratic extension field of F.*

We write $\mathfrak{X}^{\mathrm{can}}$ for the set of canonical points of $\mathfrak{X}$, and $\mathfrak{X}^E \subset \mathfrak{X}^{\mathrm{can}}$ for the set of canonical lifts admitting endomorphisms by a particular (isomorphism class of) E. Then $\mathfrak{X}^E$ forms a single orbit under the action of $\mathrm{GL}_2(F) \times B^\times$. Suppose $\mathscr{F}_E/\hat{E}^{\mathrm{nr}}$ is a particular one-dimensional formal $\mathcal{O}_E$-module of $\mathcal{O}_E$-height 1; then $\mathscr{F}_E$ is unique up to isomorphism by classical Lubin-Tate theory. The set of quasi-isogenies $\iota \colon \Sigma \to \mathscr{F}_E \otimes k^{\mathrm{ac}}$ is in bijection with the set of embeddings $E \hookrightarrow B$. Similarly, the set of isomorphisms $F^{\oplus 2} \to V(\mathscr{F})$ is in bijection with the set of embeddings $E \hookrightarrow M_2(F)$. Given $x \in \mathfrak{X}^E$, let $j_{x,M_2(F)}$ and $j_{x,B}$ be the corresponding embeddings of E into $M_2(F)$ and B, and let $j_x = j_{x,M_2(F)} \times j_{x,B} \colon E \hookrightarrow M_2(F) \times B$ be the product.

We write E_n for the field obtained from $\hat{E}^{\mathrm{nr}}$ by adjoining the π_E^n-torsion in $\mathscr{F}_E$. The action of Galois on $\mathscr{F}_E[\pi^n]$ gives an isomorphism from $\mathrm{Gal}(E_n/\hat{E}^{\mathrm{nr}})$ onto $(\mathcal{O}_E/\pi_E^n\mathcal{O}_E)^\times$.

If $\alpha \in E^\times$, then $j_x(\alpha)$ fixes the point $x \in \mathfrak{X}(\pi^n)$. We will need the following approximation for the derivative of $j_x(\alpha)$:

Lemma 3.6. *Let $n \geqslant 0$, and let $\chi \colon E^\times \to (k^{\mathrm{ac}})^\times$ be the character giving the action of $j_x(\alpha)$ on the tangent space of x in $\mathfrak{X}(\pi^n)$, taken modulo $\mathfrak{p}_E$. Then $\chi(F^\times) = 1$, and*

(i) If E/F is unramified and $\alpha \in \mathcal{O}_E^\times$, then $\chi(\alpha) \equiv \alpha^{q-1} \pmod{\pi}$.

(ii) If E/F is ramified, then $\chi(\mathcal{O}_E^\times) = 1$ and $\chi(\pi_E) = -1$ for any uniformizer π_E.

Proof. Since $\mathfrak{X}(\pi^n) \to \mathfrak{X}(1)$ is étale, we may immediately reduce to the case that $n = 0$, where it is a straightforward calculation. $\square$

3.12 The too-supersingular region

DEFINITION 3.7. The *too-supersingular disc* $\mathscr{D}^{\mathrm{ts}} \subset \mathfrak{X}(1)$ is the disc $\left\{|u| \leqslant |\pi|^{q/(q+1)}\right\}$.

If $z = (\mathscr{F}, \iota) \in \mathfrak{X}(1)$ then z belongs to $\mathscr{D}^{\mathrm{ts}}$ if and only if $\mathscr{F}[\pi]$ has no canonical subgroup.

Now suppose E/F is the unramified extension. For any $x = (\mathscr{F}, \iota, \phi) \in \mathfrak{X}(\pi^\infty)^E$ with $\mathrm{End}\,\mathscr{F} = \mathcal{O}_E$, then certainly the image of x in $\mathfrak{X}(1)$ lies in $\mathscr{D}^{\mathrm{ts}}$. We define the affinoid $\mathfrak{Z}_{x,0}$ to be the preimage of $\mathscr{D}^{\mathrm{ts}}$ in $\mathfrak{X}(\pi)$. We extend this definition to all $x \in \mathfrak{X}^E$ by the rule

$$\mathfrak{Z}_{x^g,0} = \mathfrak{Z}_{x,0}^g \subset \mathfrak{X}(g\,\mathrm{GL}_2(\mathcal{O}_F)g^{-1}),$$

all $g = (g_1, g_2) \in \mathrm{GL}_2(F) \times B^\times$.

PROPOSITION 3.8. *The stabilizer of $\mathfrak{Z}_{x,0}$ in $\mathrm{GL}_2(F) \times B^\times$ is $j_x(E^\times)(\mathrm{GL}_2(\mathcal{O}_F) \times \mathcal{O}_B^\times)$.*

THEOREM 3.9. *The affinoid $\mathfrak{Z}_{x,0}$ has reduction equal to the nonsingular affine curve with equation $(Y_1 Y_2^q - Y_1 Y_2^q)^{q-1} = 1$. The action of $\mathrm{GL}_2(\mathcal{O}_F)$ on $\overline{\mathfrak{Z}}_{x,0}$ is inflated from the action of $\mathrm{GL}_2(k)$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(Y_1, Y_2) = (aY_1 + cY_2, bY_1 + dY_2)$. The action of $\mathcal{O}_B^\times$ is inflated from the action of $k_E^\times$ via $\alpha(Y_1, Y_2) = (\alpha^{-1}Y_1, \alpha^{-1}Y_2)$. Finally the action of $j_x(\pi)$ on $\overline{\mathfrak{Z}}_{x,0}$ is trivial.*

Proof. This is a specialization to the case of height 2 of Prop. 6.15 in [Yos10]. $\square$

3.13 The ramified circle

Let $\Gamma_0(\pi) = \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F^\times \end{pmatrix}$ and $\Gamma_1(\pi) = \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O}_F \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix}$. Note that $X_1^{(1)}$ is a well-defined coordinate on $\mathfrak{X}(\Gamma_1(\pi))$.

Let E/F be a ramified quadratic extension. Suppose that $x \in \mathfrak{X}(\Gamma_1(\pi))^E$ is such that $\mathrm{End}\,\mathscr{F} = \mathcal{O}_E$, with $X_1(x)$ belonging to $\mathscr{F}[\pi_E]$.

DEFINITION 3.10. *The ramified depth zero affinoid $\mathfrak{Z}_{x,0}$ is the affinoid subdomain of $\mathfrak{X}(\Gamma_1(\pi))$ consisting of those points $z = (\mathscr{F}, \iota, \phi)$ for which $v(u(z)) = 1/2$ and for which $X_1^{(1)}(z) = \phi(1,0)$ belongs to the canonical subgroup of $\mathscr{F}_z[\pi]$.*

PROPOSITION 3.11. *$\mathfrak{Z}_{x,0}$ is isomorphic to a circle. Its stabilizer in $\mathrm{GL}_2(F) \times B^\times$ is $j_x(E^\times)(\Gamma_0(\pi) \times \mathcal{O}_B^\times)$.*

4. Bushnell-Kutzko types for $\mathrm{GL}_2(F)$ and $B^\times$, and the Jacquet-Langlands correspondence

In this section we review the constructions of Bushnell-Kutzko [BK93] regarding the classification of admissible representations of $\mathrm{GL}_n(F)$ via "strata". A stratum is (more or less) a character of a certain compact subgroup of $\mathrm{GL}_n(F)$; admissible representations may be distinguished according to which strata they contain. We borrow the notational conventions from §4 in the book [BH06], which is well-suited to the study of *supercuspidal* representations of $\mathrm{GL}_2(F)$. There is a parallel study of strata for the quaternionic unit group $B^\times$, which we also review. Finally we present results from [Wei10], wherein for each "simple" stratum S we construct a "linking orders" $\mathcal{L} \subset$

$M_2(F) \times B$ and a finite-dimensional character ρ_S of $\mathcal{L}^\times$. Loosely speaking, when ρ is induced up to $\mathrm{GL}_2(F) \times B^\times$, the result is a direct sum of representations of the form $\Pi \otimes \mathrm{JL}(\check{\Pi})$, where Π ranges over those supercuspidal representations of $\mathrm{GL}_2(F)$ which contain S. As S varies, one sees all of the wild supercuspidal representations Π. This observation is going to be crucial in our construction of a semistable covering of the Lubin-Tate tower.

4.1 Chain orders and strata

A *lattice chain* is an F-stable family of lattices $\Lambda = \{L_i\}$, with each $L_i \subset F \oplus F$ an $\mathcal{O}_F$-lattice and $L_{i+1} \subset L_i$ for all $i \in \mathbf{Z}$. There is a unique integer $e(\Lambda) \in \{1, 2\}$ for which $\pi_F L_i = L_{i+e(\Lambda)}$. Let $\mathfrak{A}_\Lambda$ be the stabilizer in $M_2(F)$ of Λ. Up to conjugacy by $\mathrm{GL}_2(F)$ we have

$$\mathfrak{A}_\Lambda = \begin{cases} M_2(\mathcal{O}_F), & e_\Lambda = 1, \\ \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}, & e_\Lambda = 2 \end{cases}$$

DEFINITION 4.1. A *chain order* in $M_2(F)$ is an $\mathcal{O}_F$-order $\mathfrak{A} \subset M_2(F)$ which is equal to $\mathfrak{A}_\Lambda$ for some lattice chain Λ. We say $\mathfrak{A}$ is unramified or ramified as e_Λ is 1 or 2, respectively.

Suppose $\mathfrak{A}$ is a chain order in $M_2(F)$; let $\mathfrak{P}$ be its Jacobson radical. Then $\mathfrak{P} = \pi \mathfrak{A}$ if $\mathfrak{A}$ is unramified and $\mathfrak{P} = \begin{pmatrix} \mathfrak{p}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix}$ in the case that $\mathfrak{A} = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}$. We have a filtration of $\mathfrak{A}^\times$ by subgroups $U_\mathfrak{A}^n = 1 + \mathfrak{P}^n$, $n \geqslant 1$.

These constructions have obvious analogues in the quaternion algebra B: If $\mathfrak{A} = \mathcal{O}_B$ is the maximal order in B, then the Jacobson radical $\mathfrak{P}$ is the unique maximal two-sided ideal of $\mathfrak{A}$, generated by a prime element π_B. We let $U_\mathfrak{A}^n = 1 + \mathfrak{P}^n$.

4.2 Characters and Bushnell-Kutzko types

In the following discussion, $\psi \colon F \to \mathbf{C}^\times$ is a fixed additive character. We assume that ψ is of level one, which means that $\psi(\mathfrak{p}_F)$ is trivial but $\psi(\mathcal{O}_F)$ is not. (The choice of level of ψ is essentially arbitrary, but it has become customary to use characters of level one.) Write $\psi_{M_2(F)}$ for the (additive) character of $M_2(F)$ defined by $\psi_{M_2(F)}(x) = \psi(\mathrm{Tr}\, x)$. Similarly define $\psi_B \colon B \to \mathbf{C}$ by $\psi_B(x) = \psi(\mathrm{Tr}_{B/F}(x))$, where $\mathrm{Tr}_{B/F}$ is the reduced trace.

Now let A be either $M_2(F)$ or B. Let $\mathfrak{A} \subset A$ be an $\mathcal{O}_F$-order which equals a chain order (if $A = M_2(F)$) or the maximal order in B (if $A = B$). Let $n \geqslant 1$. We have a character ψ_α of $U_\mathfrak{A}^m$ defined by

$$U^n/U^{n+1} \to \mathbf{C}^\times$$
$$1 + x \mapsto \psi_\alpha(\alpha x)$$

If π is an admissible irreducible representation of $\mathrm{GL}_2(F)$, one may ask for which α is the character ψ contained in $\pi|_{U^n}$. This is the basis for the classification of representations by Bushnell-Kutzko types, c.f. [BK93].

DEFINITION 4.2. A *stratum* in A is a triple of the form $S = (\mathfrak{A}, n, \alpha)$, where $n \geqslant 1$ and $\alpha \in \mathfrak{P}_\mathfrak{A}^{-n}$. Two strata $(\mathfrak{A}, n, \alpha)$ and $(\mathfrak{A}, n, \alpha')$ are equivalent if $\alpha \equiv \alpha' \pmod{\mathfrak{P}^{1-n}}$.

DEFINITION 4.3. Let $S = (\mathfrak{A}, n, \alpha)$ be a stratum.

(i) S is *ramified simple* if $E = F(\alpha)$ is a ramified quadratic extension field of F, n is odd, and $\alpha \in E$ has valuation exactly $-n$.

(ii) S is *unramified simple* if $E = F(\alpha)$ is an unramified quadratic extension field of F, $\alpha \in E$ has valuation exactly $-n$, and the minimal polynomial of $\pi^n \alpha$ over F is irreducible mod π.

(iii) S is *simple* if it is ramified simple or unramified simple.

If $S = (\mathfrak{A}, n, \alpha)$ is a stratum in $M_2(F)$ (resp., B) and Π is an admissible representation of $\mathrm{GL}_2(F)$ (resp., smooth representation of $B^\times$), we say that Π *contains the stratum* S if $\pi|_{U_{\mathfrak{A}}^n}$ contains the character ψ_α.

We call Π *minimal* if its conductor cannot be lowered by twisting by one-dimensional characters of $F^\times$.

From [BH06] we have the following classification of supercuspidal representations of $\mathrm{GL}_2(F)$:

THEOREM 4.4. *A minimal irreducible admissible representation Π of $\mathrm{GL}_2(F)$ is supercuspidal if and only if one of the following holds:*

(i) Π *contains the trivial character of $U^1_{M_2(\mathcal{O}_F)}$ (i.e. Π has "depth zero").*

(ii) Π *contains a simple stratum.*

The analogous statement for B is:

THEOREM 4.5. *A minimal irreducible representation Π of $B^\times$ of dimension greater than 1 satisfies exactly one of the following properties:*

Π *contains the trivial character of $U^1_{\mathcal{O}_B}$ (i.e. Π has "depth zero").*

Π *contains a simple stratum.*

4.3 Linking Orders

If $x \in \mathfrak{X}^E$ be represented by the triple $(\mathscr{F}_E, \iota, \phi)$, let $\mathfrak{A} = \mathfrak{A}_x$ be the chain order which stabilizes the lattice chain $\{j_{x,M_2(F)}(\mathfrak{p}_E^n)\}$. The order $\mathfrak{A}$ is normalized by $E^\times$, and $E \cap \mathfrak{A} = \mathcal{O}_E$.

It will be helpful to make the following abbreviations: $A_1 = M_2(F)$, $A_2 = B$. $\mathfrak{A}_1 = \mathfrak{A}$, $\mathfrak{A}_2 = \mathcal{O}_B$. Let $n \geqslant 1$ be an integer subject to the constraint that n is odd if $\mathfrak{A}$ is ramified. For $i = 1, 2$, let C_i be the orthogonal complement of $j_{x,A_i}(E)$ in A_i with respect to the standard trace pairing.

The *linking order* $\mathcal{L}_n = \mathcal{L}_{x,n}$ is defined by

$$\mathcal{L}_n = j_x(\mathcal{O}_E) + (\mathfrak{p}_E^n \times \mathfrak{p}_E^n) + \mathbf{V}^n,$$

where $\mathbf{V}^n = V_1^n \times V_2^n \subset \mathfrak{A} \times \mathcal{O}_B$ is a certain $\mathcal{O}_E$-submodule chosen to be large as possible subject to the constraint that $\mathcal{L}_S$ be closed under multiplication. The value of V_i^n is:

$$V_i^n = \mathfrak{p}_E^m \left(C_i \cap \mathfrak{A}_i \right), \text{ where}$$
$$m = \begin{cases} \lfloor n/2 \rfloor, & i = 2 \text{ and } E/F \text{ unramified} \\ \lfloor (n+1)/2 \rfloor, & \text{all other cases.} \end{cases}$$

Let $\mathcal{L}_S', \mathcal{L}_S^\circ \subset \mathcal{L}_S$ be the $\mathcal{O}_E$-submodules

$$\mathcal{L}_n' = j_x(\mathcal{O}_E) + \mathfrak{p}_E^n \times \mathfrak{p}_E^n + \mathbf{V}^{n+1}.$$
$$\mathcal{L}_n^\circ = j_x(\mathfrak{p}_E) + \mathfrak{p}_E^{n+1} \times \mathfrak{p}_E^{n+1} + \mathbf{V}^{n+1}.$$

Then $\mathcal{L}_n^\circ$ is a double-sided ideal of $\mathcal{L}_S$. The quotient $\mathcal{L}_n/\mathcal{L}_n^\circ$ is a k_E-module whose isomorphism class is described in [Wei10], Prop. 4.3.4.

If E/F is unramified, then

$$\mathcal{L}_n/\mathcal{L}_n^\circ \cong \left\{ \begin{pmatrix} a & b & c \\ & a^q & b^q \\ & & a \end{pmatrix} \bigg| \ a, b, c \in k_E \right\} \tag{4.3.1}$$

If E/F is ramified, then

$$\mathcal{L}_n/\mathcal{L}_n^\circ \cong \left\{ \begin{pmatrix} a & & c \\ & a & \\ & & a \end{pmatrix} \bigg| \ a, c \in k \right\} \tag{4.3.2}$$

In each case the center of $(\mathcal{L}_n/\mathcal{L}_n^\circ)^\times$ is $(\mathcal{L}_n'/\mathcal{L}_n^\circ)^\times$; the quotient is abelian.

Define a group $\mathcal{K} = \mathcal{K}_n \subset \mathrm{GL}_2(F) \times B^\times$ by

$$\mathcal{K} = j_x(E^\times)\mathcal{L}_n^\times.$$

Similarly set $\mathcal{K}' = j_x(E^\times)(\mathcal{L}_n')^\times$.

Assume that q is odd, and let Z/k^{ac} be the nonsingular projective curve with affine equation

$$Z := \begin{cases} Y^{q^2} - Y = X^{q^2+q} - X^{q+1}, & E/F \text{ unramified} \\ Y^q - Y = X^2, & E/F \text{ ramified} \end{cases}$$

Define an action of $\mathcal{K}$ on Z by the conditions

(i) The action of $\mathcal{L}_n^\times$ on Z is inflated from the action of $(\mathcal{L}_S/\mathcal{L}_S^\circ)^\times$ on Z given by the presentation of $(\mathcal{L}_n/\mathcal{L}_n^\circ)^\times$ as a subgroup of $\mathrm{PGL}_3(k_E)$ as in Eqs. (4.3.1) and (4.3.2).

(ii) If S is unramified, then the action of $\Delta(\pi)$ on Z is trivial.

(iii) If S is ramified, then the action of $\Delta(E^\times)$ on Z factors through $\Delta(E^\times/\mathcal{O}_E^\times)$, and $\Delta(\pi)(X, Y) = (X, -Y)$.

Then $H^1(Z, \mathbf{Q}_\ell^{\mathrm{ac}}) \cong \bigoplus_S \rho_S$ decomposes as a direct sum of $\mathcal{K}$-modules ρ_S, one for each equivalence class of simple stratum S of the form $(\mathfrak{A}, n, \alpha)$. The restriction of ρ_S to $U_{\mathfrak{A}}^n$ is a direct sum of the characters ψ_α from §4.2.

The following paraphrases Thm. 6.0.1 of [Wei10]:

THEOREM 4.6. *Let Π (resp., Π') be a minimal supercuspidal representation of $\mathrm{GL}_2(F)$ (resp., a minimal representation of $B^\times$ of dimension > 1). Assume that the central characters of Π and Π' are contragredient to one another. The following are equivalent:*

(i) $\dim \mathrm{Hom}_{\mathcal{K}}\big(H^1(Z, \mathbf{Q}_\ell), \Pi \otimes \Pi'\big) \neq 0$.

(ii) Π *contains a simple stratum of the form* $(\mathfrak{A}, n, \beta)$, *and* $\Pi' = \mathrm{JL}(\check{\Pi})$.

4.4 Characters with values in a π-divisible group, and their "splitting" by functions on the Lubin-Tate tower

We adapt the machinery of strata to the scenario of the Lubin-Tate tower $\mathfrak{X}(\pi^n)$. We have the formal $\mathcal{O}_F$-module LT over $\hat{\mathcal{O}}_F^{\mathrm{nr}}$ of height 1 and its associated family $\mathrm{LT}[\pi^n]$ of finite group schemes. For each $r \geqslant 1$ we have a section $\Delta^{(r)}$ of $\mathrm{LT}[\pi^r]$ over $\mathfrak{X}(\pi^n)$, and these are compatible in the sense that $[\pi]_{\mathrm{LT}}(\Delta^{(r+1)}) = \Delta^{(r)}$. We can define a character ψ of $F/\mathfrak{p}_F$ along the lines of §4.2 with values not in $\mathbf{C}^\times$ but rather in the divisible $\mathcal{O}_F$-module

$$\mathrm{LT}[\pi^\infty](\mathfrak{X}(\pi^\infty)) := \varinjlim \mathrm{LT}[\pi^n](\mathfrak{X}(\pi^n)).$$

An element of $\mathrm{LT}[\pi^\infty](\mathfrak{X}(\pi^\infty))$ can be considered as an F_∞-valued locally constant function on $\mathfrak{X}(\pi^\infty)$ which happens to take values in the set of π-power division points of LT. If $a \in F$, so that $b = \pi^n a \in \mathcal{O}_F$ for some $n \geqslant 0$, write $\psi(a) = [b]_{\mathrm{LT}}(\Delta^{(n+1)})$. Clearly the definition of ψ does not depend on n. We also define $\psi_{M_2(F)} \colon M_2(F) \to \mathrm{LT}[\pi^\infty](\mathfrak{X}(\pi^\infty))$ by $\psi_A(M) = \psi(\mathrm{Tr}\, M)$.

Let $x \in \mathfrak{X}^E$ be a canonical lift, and let $\mathfrak{A} = \mathfrak{A}_x$. If $S = (\mathfrak{A}, n, j_{x,M_2(F)}(\alpha))$ is a stratum, we may define a character $\psi_\alpha \colon U_{\mathfrak{A}}^n \to \mathrm{LT}[\pi](\mathfrak{X}(\pi^\infty))$ exactly as in §4.2. Similarly, write n' to be the integer $2n$ (resp., n) as E/F is unramified (resp., ramified). Then $S' = (\mathcal{O}_B, n', j_{x,B}(\alpha))$ is a stratum in B and we have the character $\psi'_S \colon U_B^{n'} \to \mathrm{LT}[\pi^\infty](\mathfrak{X}(\pi^\infty))$.

THEOREM 4.7. *Assume either $F = \mathbf{Q}_p$ with p odd, or else that F has positive odd characteristic. Let $S = (\mathfrak{A}, n, \alpha)$ be a stratum in $M_2(F)$. There exists an integral analytic function W_α on $\mathfrak{X}(U^{n+1})_{\mathfrak{A}}$ with the following properties:*

(i) *For all $g \in U_{\mathfrak{A}}^n$:*
$$g(W_\alpha) -_{\mathrm{LT}} W_\alpha = \psi_\alpha(g)$$

(ii) *For all $b \in U_{\mathfrak{A}}^m$, the congruence*
$$b(W_\alpha) -_{\mathrm{LT}} W_\alpha \equiv -\psi'_\alpha(g) \pmod{\pi^{1/2}}$$
is valid on the ring of integral analytic functions on the inverse image of $\mathfrak{Z}_{x,0}$ in $\mathfrak{X}(U^{n+1})$.

Condition (1) means that ψ_α is "split" by W_S, in the sense that the image of ψ_α in
$$H^1(U^n, \mathrm{LT}[\pi](\mathfrak{X}(U^{n+1}))) \to H^1(U^n, \mathcal{A}(U^{n+1}))$$
is zero, and that ψ_α is the image of W_α under the boundary map.

4.5 Proof of Thm. 4.7, case of equal characteristic

Suppose $F \cong \mathbf{F}_q((\pi))$ has characteristic p; we will construct the coordinate W_S required by Thm. 4.7 manually using the model of $\mathscr{F}^{\mathrm{univ}}$ appearing in §3.8.

Here we select a model of LT which satisfies
$$X +_{\mathrm{LT}} Y = X + Y$$
$$[a]_{\mathrm{LT}}(X) = aX,\ a \in k$$
$$[\pi]_{\mathrm{LT}}(X) = \pi X - X^q.$$

Define the determinant form
$$\mu(X, Y) = \det \begin{pmatrix} X & Y \\ X^q & Y^q \end{pmatrix}. \tag{4.5.1}$$

A simple calculation shows that $\mu \colon \mathscr{F}^{\mathrm{univ}}[\pi] \times \mathscr{F}^{\mathrm{univ}}[\pi] \to \mathrm{LT}[\pi]$ identifies the top exterior power of $\mathscr{F}^{\mathrm{univ}}[\pi]$ with $\mathrm{LT}[\pi]$. A concrete realization of the function $\Delta = \Delta^{(1)}$ from §3.6 is therefore provided by $\Delta = \mu(X_1^{(1)}, X_2^{(1)})$. (For a more general construction of $\Delta^{(n)}$ for formal $\mathcal{O}_F$-modules of arbitrary height, see [Weib], Thm. 3.2.)

Let us abbreviate $X_i = X_i^{(1)}$, $i = 1, 2$. Note that, for $M \in M_2(\mathcal{O}_F)$, we have
$$\mu(M(X_1), X_2) + \mu(X_1, M(X_2)) = [\mathrm{Tr}\, M]_{\mathrm{LT}}(\Delta). \tag{4.5.2}$$

Excluding the case of x ramified and $n = 1$, we take our coordinate W_α to be
$$W_\alpha = \mu(\alpha(X_1), X_2) + \mu(X_1, \alpha(X_2)),$$

where we recall the meaning of $\alpha(X_i)$ for non-integral α from §3.10. Then part (1) of Thm. 4.7 follows directly from Eq. (4.5.2): If $g = 1 + M \in U_{\mathfrak{A}}^n$, then

$$g(W_\alpha) - W_\alpha = \mu\left(M\alpha(X_1), X_2\right) + \mu\left(X_1, M\alpha(X_2)\right) = [\mathrm{Tr}\, M\alpha]_{\mathrm{LT}}(\Delta) = \psi_\alpha(g).$$

(The assumption about x was necessary to ensure that $M \in \pi M_2(\mathcal{O}_F)$, which implies that $g(X_i) = X_i$.)

As for part (2): let $\varpi \in \mathcal{O}_{E_1}$ denote an element of valuation $q/(q+1)$ (resp., $1/2$) as x is unramified (resp., ramified). Over the domain $\mathfrak{Z}_{x,0}$, $\mathscr{F}^{\mathrm{univ}} \otimes (\mathcal{O}_{E_1}/\varpi)$ is isomorphic to $\mathscr{F}_x \otimes (\mathcal{O}_{E_1}/\varpi)$, where $\mathscr{F}_x$ is the formal $\mathcal{O}_F$-module corresponding to x; by the main theorem of [Gro86] we have $\mathrm{End}(\mathscr{F}^{\mathrm{univ}} \otimes (\mathcal{O}_{E_1}/\varpi)) = \mathcal{O}_B$. For $\beta \in \mathcal{O}_B$, write $[\beta](X) \in (\mathcal{O}_{E_1}/\varpi)$ for the power series that realizes the endomorphism of $\mathscr{F}^{\mathrm{univ}} \otimes (\mathcal{O}_{E_1}/\varpi)$ corresponding to β. By Thm. 3.2, we have $\mu([\beta](X_1), [\beta](X_2)) = [N_{B/F}(b)^{-1}]_{\mathrm{LT}}(\Delta)$ whenever β lies in $\mathcal{O}_B^\times$; one can deduce from this that

$$\mu([\beta](X_1), X_2) + \mu(X_1, [\beta](X_2)) \equiv [N_{B/F}(1+b) - 1]_{\mathrm{LT}}(\Delta) \equiv [\mathrm{Tr}_{B/F}(b)]_{\mathrm{LT}}(\Delta) \pmod{\varpi}.$$

This immediately gives part (2) of the theorem.

Now suppose x is ramified and $n = 1$. Then if $g \in U^1$, one has $g(X_1) = X_1$, but $g(X_2) = X_2 + aX_1$ for some $a \in k$. The appropriate coordinate is

$$W_\alpha = \mu\left(\alpha(X_1), X_2\right) + \mu\left(X_1, \alpha(X_2)\right) + \frac{X_2}{X_1}\mu\left(X_2, \alpha(X_2)\right).$$

4.6 Proof of Thm. 4.7, case of $F = \mathbf{Q}_p$

Take $\pi = p$. Choose a supersingular elliptic curve $A/\mathbf{F}_p^{\mathrm{ac}}$. We exploit the point of view that for a complete local $\mathbf{Z}_p^{\mathrm{nr}}$-algebra R with residue field $\mathbf{F}_p^{\mathrm{ac}}$, $\mathfrak{X}(p^n)(R)$ is the set of triples (E, ι, ϕ), where E/R is an elliptic curve and $\iota \in \mathrm{Hom}(A, E \otimes \mathbf{F}_p^{\mathrm{ac}}) \otimes \mathbf{Z}_p$ is not divisible by p (which is to say, ι induces a homomorphism on the level of formal groups). Let $(\mathcal{E}^{\mathrm{univ}}, \iota^{\mathrm{univ}}, \phi_n)$ be the universal triple over $\mathfrak{X}(p^n)$. From this perspective the coordinates $X_i^{(n)}$ are sections of $\mathcal{E}^{\mathrm{univ}}[p^n]$ over $\mathfrak{X}(\pi^n)$.

Assume $\mathrm{LT} = \mathbf{G}_m$ is the multiplicative group, and that $\Delta^{(n)} \colon \mathfrak{X}(\pi^n) \to \mathbf{G}_m[p^n]$ is the usual Weil pairing. Then part (1) of the theorem translates to the condition $g(W_\alpha)/W_\alpha = \psi_\alpha(g)$ for $g \in U^n$. We construct an analogue of the determinant form in Eq. (4.5.1) as follows. Suppose P is a section of $\mathcal{E}^{\mathrm{univ}}[p]$, and Q is a section of $\mathcal{E}^{\mathrm{univ}}[p^m]$ for some $m \geqslant 1$. Choose an arbitrary section x of $\mathcal{E}^{\mathrm{univ}}$ which does not pass anywhere through $\mathcal{E}^{\mathrm{univ}}[p^m]$. The divisor $[p]^*(P) - [p]^*(O)$ is principal; let $g \colon \mathcal{E}^{\mathrm{univ}} \to \mathbf{P}^1$ have $\mathrm{div}\, g = [p]^*(P) - [p]^*(O)$. Then define

$$\mu(P, Q) = \frac{g(x+Q)}{g(x)}.$$

If Q' is a section of $\mathcal{E}^{\mathrm{univ}}[p]$, then $\mu(P, Q') = \Delta(P, Q')$ is the Weil pairing, and in fact

$$\begin{aligned}
\mu(P, Q+Q') &= \frac{g(x+Q+Q')}{g(x)} \\
&= \frac{g(x+Q+Q')}{g(x+Q)}\frac{g(x+Q)}{g(x)} \\
&= \Delta(P, Q')\mu(P, Q).
\end{aligned}$$

If we are outside the case of x ramified and $n = 1$, then the coordinate

$$W_\alpha = \frac{\mu\left(\alpha(X_1), X_2\right)}{\mu\left(\alpha(X_2), X_1\right)}$$

satisfies the conditions of the theorem, just as in the previous section. If x is ramified and $n = 1$ then we put

$$W_\alpha = \frac{\mu\left(\alpha(X_1), X_2\right)}{\mu\left(\alpha(X_2), X_1\right)} \exp\left[\frac{\log_\mathcal{E}(X_2)}{\log_\mathcal{E}(X_1)} \mu\left(X_2, \alpha(X_2)\right)\right],$$

where $\log_\mathcal{E} : \hat{\mathcal{E}}^{\mathrm{univ}} \to \mathbf{G}_a$ is the formal logarithm and $\exp : \mathbf{G}_a \to \mathbf{G}_m$ is the usual exponential.

5. Construction of affinoids with good reduction

5.1 Generalities on good affinoids of higher genus

The following lemma, due to Robert Coleman, shows that an affinoid subdomain Z of a wide open curve U whose reduction is a nonsingular curve of high genus must (up to removal of finitely many discs) show up as the underlying affinoid of one of the wide open subspaces of any semistable covering of U.

A Zariski subaffinoid of Z is the inverse image of a Zariski open subset $\overline{Z}^\circ \subset \overline{Z}$ under the reduction map $Z \to \overline{Z}$.

PROPOSITION 5.1. *Let U be a wide open curve over a complete DVR with algebraically closed residue field, and let $Z \subset U$ be an affinoid. Assume that $\overline{Z}$ is a smooth connected curve whose projective closure $\overline{Z}^{\mathrm{cl}}$ has positive genus. Suppose $\{U_i\}_{i \in I}$ is a locally finite semistable covering of U, with Z_i an underlying affinoid in U_i. Then there exists $i \in I$ for which a Zariski subaffinoid of Z is contained in Z_i.*

Proof. We assert there exists i such that U_i meets Z but for which $Z \cap U_i$ is not contained in a union of finitely many residue classes of Z. For otherwise we have that for all $i \in I$, the intersection $Z \cap U_i$ is contained in a (possibly empty) finite union $\coprod_{j \in J_i} R_{ij}$ of residue discs R_{ij} of Z. Since the covering is locally finite, R_{ij} is nonempty for only finitely many i. But then the R_{ij} constitute a finite collection of residue discs which cover Z, contradiction.

Therefore there exists i for which $Z \cap U_i \neq \emptyset$ is not contained in a finite union of residue classes of Z. We may write $U_i = \{x \in U : |f_j(x)| < 1 \text{ and } |g_k(x)| > 1\}$ for a finite collection of functions f_j, g_k on U. For each j, we must have $\|f_j(x)\|_Z < 1$, for otherwise the subset of Z on which $|f_j(z)| < 1$ is a finite union of residue classes containing $Z \cap U_i$, which we have excluded.

By the above argument, we have that $Z \setminus U_i = \cup_k \{x \in Z : |g_k(z)| \leqslant 1\}$ so that $\overline{Z \setminus U_i}$ is a finite set of points. Therefore U_i contains a Zariski affinoid $Z^\circ \subset Z$, namely the complement in Z of $\mathrm{red}^{-1}(\overline{Z \setminus U_i})$.

We claim that Z° is a Zariski subaffinoid of the minimal underlying affinoid V of U_i. Here we apply the assertion that the reduction of Z has positive genus. Certainly $Z^\circ \cap V \neq \emptyset$, because the complement of V in U_i is a union of annuli and Z° cannot be contained in an annulus. Nor could Z° be contained in a residue disc of V. Thus $Z^\circ \setminus V$ is a finite union of open discs contained in $U_i \setminus V$, which is in turn a disjoint union of annuli: Since each such disc is connected to $Z^\circ \cap V$, we conclude in fact that $Z^\circ \setminus V = \emptyset$. Thus Z° is a Zariski subaffinoid of V as required. $\square$

COROLLARY 5.2. *Suppose U is a wide open curve admitting a semistable covering whose dual graph is contractible. Let $Z \subset U$ be an affinoid for which $\overline{Z}^{\mathrm{cl}}$ is a nonsingular curve of positive genus. Then the restriction map*

$$H_c^1(U, \mathbf{Q}_\ell) \to H_c^1(Z, \mathbf{Q}_\ell) \to H^1(\overline{Z}^{\mathrm{cl}}, \mathbf{Q}_\ell)$$

is a surjection.

Proof. Follows immediately from Prop. 2.7 and Prop. 5.1. $\qquad\square$

5.2 The affinoids $\mathfrak{Z}_{x,n}$, $n \geqslant 1$

Define subgroups $K_{x,n}, K'_{x,n} \subset \mathfrak{A}^\times$ by $K_{x,n} = \mathcal{K}_{x,n} \cap \mathrm{GL}_2(F)$ and $K'_{x,n} = \mathcal{K}'_{x,n} \cap \mathrm{GL}_2(F)$. Similarly, define subgroups $K^B_{x,n}, (K^B_{x,n})' \subset \mathcal{O}^\times_B$ by $K^B_{x,n} = \mathcal{K}_{x,n} \cap B^\times$ and $(K^B_{x,n})' = \mathcal{K}'_{x,n} \cap B^\times$. The character ψ_α of §4.4 is well-defined on $K'_{x,n}$ and on $(K^B_{x,n})'$. Assume that we are in the setting of Thm. 4.7, so that for each $\alpha \in \mathfrak{p}^{-n}_E$ we may select a function W_α on $\mathfrak{X}(U^{n+1})$ which splits ψ_α. Then W_α is well-defined on $K_{x,n+1}$.

We define a family of affinoids $\mathfrak{Z}_{x,n}$ for $n \geqslant 1$ recursively as follows:

DEFINITION 5.3. Suppose $n \geqslant 1$. The affinoid $\mathfrak{Z}_{x,n} \subset \mathfrak{X}(K_{x,n+1})$ consists of those z for which

(i) The image of z under $\mathfrak{X}(K_{x,n}) \to \mathfrak{X}(K_{x,n-1})$ lies in the residue disc of x in $\mathfrak{Z}_{x,n}$.

(ii) For all $\alpha \in E$ for which $v_E(\alpha) = -n$, we have $|W_\alpha(z) -_{\mathrm{LT}} W_\alpha(x)| \leqslant |\Delta(x)| = |\pi|^{1/(q-1)}$.

It is not a priori clear that $\mathfrak{Z}_{x,n}$ is well-defined, because "the residue disc of x" does not make sense if $\mathfrak{Z}_{x,n-1}$ is not an affinoid. This will be established in part (i) of the next theorem. If f is a rigid analytic function on $\mathfrak{X}(K)$, $K' \subset K$ is a subgroup, and $\mathfrak{Z} \subset \mathfrak{X}(K')$ is an affinoid, then by $\|f\|_{\mathfrak{Z}}$ we mean the (Gauss) norm of f on the preimage of $\mathfrak{Z}$ in $\mathfrak{X}(K')$.

THEOREM 5.4. *Let* $n \geqslant 1$.

(i) *If* $\alpha \in E$ *has* $v_E(\alpha) = -n$, *then* $\|W_\alpha -_{\mathrm{LT}} W_\alpha(x)\|_{\mathfrak{Z}_{x,n-1}} > |\Delta(x)|$.

(ii) *The stabilizer of* $\mathfrak{Z}_{x,n}$ *in* $\mathrm{GL}_2(F) \times B^\times$ *is* $\mathcal{K}_{x,n}$.

(iii) *If* E/F *is unramified,* $\mathfrak{Z}_{x,n}$ *has good reduction isomorphic to the nonsingular affine curve over* k^{ac} *with equation*
$$Y^{q^2} - Y = X^{q^2+q} - X^{q+1}.$$

(iv) *If* E/F *is ramified and* n *is even,* $\mathfrak{Z}_{x,n}$ *has reduction isomorphic to the nonsingular affine curve over* k^{ac} *with equation*
$$Y^q - Y = X^2.$$

(v) *If* E/F *is ramified and* n *is odd,* $\mathfrak{Z}_{x,n}$ *has reduction isomorphic to a disjoint union of copies of the affine line.*

(vi) *The action of* $\mathcal{K}_{x,n}$ *on* $\overline{\mathfrak{Z}}_{x,n}$ *factors through the action of* $\mathcal{K}_{x,n}/\mathcal{K}_{x,n+1}$ *as described in* §4.3.

Proof. Throughout the proof, $n \geqslant 1$ will be fixed, and if $n \geqslant 2$, then parts (i)-(vi) will be assumed true for $n-1$.

For part (i): Assume there exists $\alpha \in E$ of valuation $-n$ with $\|W_\alpha - W_\alpha(x)\|_{\mathfrak{Z}_{x,n-1}} \leqslant |\Delta(x)|$. Let $K = K'_{x,n} \cap \psi_\alpha$, so that W_α is well defined on $\mathfrak{X}(K)$. Let $\mathfrak{Z}_K$ be the inverse image of $\mathfrak{Z}_{x,n-1}$ in $\mathfrak{X}(K)$. Then $\overline{\mathfrak{Z}}_K \to \overline{\mathfrak{Z}}_{x,n-1}$ is an Artin-Schreier cover with equation $Y^q - Y = f$, for some f in the affine coordinate ring of $\overline{\mathfrak{Z}}_{x,n-1}$. Here the coordinate Y on $\overline{\mathfrak{Z}}_K$ is the reduction of the coordinate
$$Y_\alpha = \frac{W_\alpha -_{\mathrm{LT}} W_\alpha(x)}{|\Delta^{(1)}(x)|} \tag{5.2.1}$$

on $\mathfrak{Z}_K$; this is integral by hypothesis. The action of $K'_{x,n}$ on $\overline{\mathfrak{Z}}_K$ is the Artin-Schrier action; we have $K_{x,n}/K' \cong \mathrm{Gal}(\overline{\mathfrak{Z}}_K/\overline{\mathfrak{Z}}_{x,n-1})$.

Since the group $K^B_{x,n-1}$ stabilizes $\mathfrak{Z}_{x,n-1}$, it stabilizes $\mathfrak{Z}_K$ as well. The subgroup $(K^B_{x,n})'$ acts as the identity on $\overline{\mathfrak{Z}}_{x,n}$ and acts on $\overline{\mathfrak{Z}}_K$ through the Artin-Schreier action. If n is odd, then $(K^B_{x,n})' \subsetneq$

$K_{x,n}^B$, and the latter group $K_{x,n}^B$ acts trivially on $\overline{\mathfrak{Z}}_{x,n-1}$ by the inductive hypothesis. Therefore the Artin-Schrier action $(K_{x,n}^B)' \to \mathrm{Gal}(\overline{\mathfrak{Z}}_K/\overline{\mathfrak{Z}}_{x,n-1}) \approx \mathbf{F}_p$ extends to $K_{x,n}^B$; this is impossible by inspection. On the other hand suppose n is even. Let H be the kernel of $K_{x,n-1}^B \to \mathrm{Aut}\,\overline{\mathfrak{Z}}_K$; then $H \subset K_{x,n-1}^B$ is a normal subgroup for which $H \cap (K_{x,n}^B)' = K$; this too is impossible by inspection.

We now explain why part (i) shows that the region $\mathfrak{Z}_{x,n}$ of Defn. 5.3 is an affinoid. Let $\tilde{\mathfrak{Z}}_{x,n}$ denote the affinoid subdomain of the inverse image of $\mathfrak{Z}_{x,n-1}$ in $\mathfrak{X}(K_{x,n})$ defined by the conditions $|W_\alpha(z) -_{\mathrm{LT}} W_\alpha(x)| \leqslant |\Delta(x)|$ for all α with $v_E(\alpha) = -n$. Then certainly x has image in $\mathfrak{X}(K_{x,n+1})$ belonging to $\mathfrak{Z}_{x,n}$. Part (i) and the maximum modulus principle imply that the image of $\tilde{\mathfrak{Z}}_{x,n}$ in $\mathfrak{Z}_{x,n}$ is contained in the union of finitely many residue discs. Now we see that $\mathfrak{Z}_{x,n}$ really is an affinoid: it is merely the union of those connected components of $\tilde{\mathfrak{Z}}_{x,n}$ whose image in $\mathfrak{Z}_{x,n-1}$ lie in the residue disc of x.

We turn to the remaining parts of the theorem. By the inductive hypothesis, the stabilizer of $\mathfrak{Z}_{x,n-1}$ is $\mathcal{K}_{x,n-1}$ (when $n = 1$ this is Prop. 3.8). Within $\mathcal{K}_{x,n-1}$, the stabilizer of the image of x in $\overline{\mathfrak{Z}}_{x,n-1}$ is $\mathcal{K}_{x,n}$. Let $\mathcal{K}$ be the stabilizer of $\mathfrak{Z}_{x,n}$, so that $\mathcal{K} \subset \mathcal{K}_{x,n}$. On the other hand, from the defining property of the variables W_S, we have $\mathcal{K}'_{x,n} \subset \mathcal{K}$. Let Y_α be the coordinate defined in Eq. (5.2.1). For $g \in U^n$, let $\overline{\psi}_\alpha(g)$ be the image of $\psi_\alpha(g)/\Delta(x)$ in k^{ac}, so that $g(Y_\alpha) = Y_\alpha + \overline{\psi}_\alpha(g)$. Similarly, $b(Y_\alpha) = Y_\alpha + \overline{\psi}_\alpha^B(b)$ for $b \in U_B^{n'}$. The association $\alpha \mapsto Y_\alpha$ is a k-linear map from $\mathfrak{p}_E^{-n}/\mathfrak{p}_E^{-n+1}$ to the coordinate ring of $\overline{\mathfrak{Z}}_{x,n}$. Our analysis proceeds case by case.

Case (1): E/F is unramified, and n is odd. Then $K'_{x,n} = K_{x,n}$. The image of $\mathfrak{Z}_{x,n}$ in $\mathfrak{Z}_{x,n-1}$ is a disc containing x; let X be a coordinate on this disc centered around x, so that $\|X\|_{\mathfrak{Z}_{x,n}} = 1$ and $X(x) = 0$. Passing to $\overline{\mathfrak{Z}}_{x,n}$, we have that $Y_\alpha^q - Y_\alpha = f_\alpha(X)$ for some $f_\alpha(X) \in k^{\mathrm{ac}}[X]$ with no constant term. In fact $\overline{\mathfrak{Z}}_{x,n}$ is defined by the system of Artin-Schreier equations $Y_\alpha^q - Y_\alpha = f_\alpha(X)$ as α ranges through a k-basis for $\mathfrak{p}_E^{-n}/\mathfrak{p}_E^{1-n}$; in particular $\overline{\mathfrak{Z}}_{x,n}$ is nonsingular.

Let $\zeta \in \mathcal{O}_E^\times$ be a primitive root of unity of order $q^2 - 1$. The element $j_x(\zeta)$ induces an automorphism $[\zeta]$ on $\overline{\mathfrak{Z}}_{x,n}$; we have by Prop. 3.6 that $[\zeta](X) = \zeta^{q-1}X$. The difference $[\zeta](Y_\alpha) - Y_\alpha$ is invariant under the action of $1 + \mathfrak{p}_E^n$, so that $[\zeta](Y_\alpha) - Y_\alpha = h$ for some polynomial $h_\alpha(X)$; since ζ has order $(q^2 - 1)$ we must have $h(X) + h(\zeta^{q-1}X) + \cdots + h(\zeta^{(q-1)(q^2-2)}X) = 0$, which shows that h cannot have any terms of degree divisible by $q + 1$. This means exactly that there exists $r(X)$ for which $h(X) = r(\zeta^{q-1}X) - r(X)$. Then $Y_\alpha - r(X)$ is *invariant* under $[\zeta]$. We will rename this new variable Y_α; it still has the property that $g(Y_\alpha) = Y_\alpha + \overline{\psi}_\alpha(g)$ for $g \in U^n$.

Recall that $Y_\alpha^q - Y_\alpha$ is a polynomial in X; this polynomial must be invariant under $[\zeta]$, which is to say that $Y_\alpha^q - Y_\alpha = g_\alpha(X^{q+1})$ for some polynomial g_α. There must exist $\alpha \in \mathfrak{p}_E^{-n}$ with $g_\alpha \neq 0$; otherwise we would have $\|W_\alpha - W_\alpha(x)\|_{\mathfrak{Z}_{x,n}} < 1$ for all such α. These considerations show that $\overline{\mathfrak{Z}}_{x,n}^{\mathrm{cl}}$ is a (possibly disconnected) nonsingular projective curve of positive genus. Thus $H^1(\overline{\mathfrak{Z}}_{x,n}^{\mathrm{cl}}, \mathbf{Q}_\ell^{\mathrm{ac}}) \neq 0$. A calculation with the Lefschetz fixed-point formula shows the $\mathcal{K}'_{x,n}$-module $H^1(\overline{\mathfrak{Z}}_{x,n}, \mathbf{Q}_\ell^{\mathrm{ac}})$ contains (the ℓ-adic version of) ψ_α if and only if $g_\alpha \neq 0$, in which case it must appear with multiplicity at least q.

By Cor. 5.2, $H_c^1(\mathfrak{X}, \mathbf{Q}_\ell)$ surjects onto $\mathrm{Ind}_{\mathcal{K}}^{\mathrm{GL}_2(F) \times B^\times} H^1(\overline{\mathfrak{Z}}_{x,n}^{\mathrm{cl}}, \mathbf{Q}_\ell)$. This shows that $H^1(\overline{\mathfrak{Z}}_{x,n}^{\mathrm{cl}}, \mathbf{Q}_\ell^{\mathrm{ac}})$ may contain only those ψ_α for which $(\mathcal{A}, n, \alpha)$ is simple. For each simple stratum $(\mathfrak{A}, n, \alpha)$, there is a unique q-dimensional character ρ_S of $\mathcal{K}_{x,n}$ lying above ψ_α. The $\mathcal{K}_{x,n}$-module $\mathrm{Ind}_{\mathcal{K}}^{\mathcal{K}_{x,n}} H^1(\overline{\mathfrak{Z}}_{x,n}^{\mathrm{cl}}, \mathbf{Q}_\ell^{\mathrm{ac}})$ must decompose as a sum over ρ_S with multiplicity one. The only possibility is $\mathcal{K} = \mathcal{K}_{x,n}$, and $H^1(\overline{\mathfrak{Z}}_{x,n}, \mathbf{Q}_\ell^{\mathrm{ac}}) = \bigoplus_S \rho_S$, where S runs over simple strata of the form $(\mathfrak{A}, n, \alpha)$.

We are now in a position to derive an equation for $\overline{\mathfrak{Z}}_{x,n}$. We must have $\deg g_\alpha = 1$ for each

α with $(\mathfrak{A}, n, \alpha)$ simple, and $g_\alpha = 0$ whenever $\overline{\pi^n \alpha}$ belongs to k. After rescaling X we have $g_{\pi^{-n}\zeta} = (\zeta^q - \zeta)X^{q+1}$. Let $\{\zeta, \zeta^q\}$ be a basis for k_E/k, and let $\beta \in k_E$ be an element satisfying

$$\beta\zeta + \beta^q\zeta^q = 1 \tag{5.2.2}$$
$$\beta\zeta^q + \beta^q\zeta = 0. \tag{5.2.3}$$

Define a coordinate W on $\mathfrak{Z}_{x,n}$ by

$$W = \beta Y_{\pi^{-n}\zeta} + \beta^q Y_{\pi^{-n}\zeta^q}.$$

Then $W^{q^2} - W = X^{q^2+q} - X^{q+1}$ furnishes a single equation for $\overline{\mathfrak{Z}}_{x,n}$.

Case (2): E/F is unramified, and n is even. Then $K'_{x,n} \subsetneq K_{x,n}$. Let $\mathfrak{Z}'_{x,n}$ be the image of $\mathfrak{Z}_{x,n}$ in $\mathfrak{X}(K'_{x,n})$. We claim that $\mathfrak{Z}'_{x,n}$ is a disjoint union of discs. Let $\{W_i\}$ be a $K_{x,n}$-equivariant covering of $\mathfrak{X}(K_{x,n+1})$. Refine $\{W_i\}$ so that whenever Z_i is an underlying affinoid of W_i which meets $\mathfrak{Z}_{x,n}$, we have $Z_i \subset \mathfrak{Z}_{x,n}$. Let Z'_i be the image of Z_i in $\mathfrak{Z}'_{x,n}$. By the same reasoning as in case (1), $\overline{Z}_i \to \overline{Z}'_i$ is an Artin-Schreier cover. Suppose $\mathcal{K}_i$ is the stabilizer of $\overline{Z}_i$; then $\mathcal{K}_i$ contains $\mathcal{K}'_{x,n}$. As a $K'_{x,n}$-module we have

$$[H^*(\overline{Z}_i^{\mathrm{cl}}, \mathbf{Q}_\ell)] = \chi\left(\overline{Z}_i^{',\mathrm{cl}}\right) \mathbf{Q}_\ell[K'_{x,n}/K_{x,n+1}] + [V],$$

where V is a $K'_{x,n}/K_{x,n+1}$-module which does not contain the trivial character. The module $H^1(\overline{Z}_i^{\mathrm{cl}}, \mathbf{Q}_\ell)$ cannot contain the trivial character of $K'_{x,n}/K_{x,n+1}$, for this would result in non-supercuspidal representations of $\mathrm{GL}_2(F)$ appearing in $H^1_c(\mathfrak{X}, \mathbf{Q}_\ell^{\mathrm{ac}})$; one quickly sees that $H^1(\overline{Z}_i^{',\mathrm{cl}}, \mathbf{Q}_\ell) = 0$ and that $\overline{Z}'_i$ is an open subset of the affine line. Each Z_i is therefore a disc with (possibly) finitely many open discs removed. Since $Z'_{x,n}$ is covered by wide opens W_i whose underlying affinoids are these Z_i, it must be that $Z'_{x,n}$ is a disjoint union of discs.

Let $\mathfrak{Z}''_{x,n}$ be the connected component of $\mathfrak{Z}'_{x,n}$ which contains x. Let X be a normalized coordinate on $\mathfrak{Z}''_{x,n}$, so that $X(x) = 0$ and $\|X\|_{\mathfrak{Z}''_{x,n}} = 1$. Then as in the previous paragraph, $Y_\alpha^q - Y_\alpha = g(X^{q+1})$ on $\mathfrak{Z}''_{x,n}$. One now argues exactly as in the previous paragraph: the stabilizer of $\mathfrak{Z}''_{x,n}$ must be $\mathcal{K}_{x,n}$, which shows that in fact $\mathfrak{Z}'_{x,n} = \mathfrak{Z}''_{x,n}$ is a single disc.

Case (3): x is ramified and n is odd. Then $\mathcal{K}'_{x,n} = \mathcal{K}_{x,n}$, which already establishes part ii. This time the action of $j_x(E^\times)$ on the tangent space of x is through a character of order 2, so that the equation for the reduction of $\mathfrak{Z}_{x,n}$ takes the form $Y^q - Y = g(X^2)$, where $Y = Y_\alpha$ and α is any element of E with $v_E(\alpha) = -n$. This ensures that $H^1(\overline{\mathfrak{Z}}_{x,n}^{\mathrm{cl}}, \mathbf{Q}_\ell)$ contains ψ_S for each simple $S = (\mathfrak{A}_x, n, \alpha)$ with multiplicity at least one. In fact the multiplicity must be exactly one, since by Cor. 5.2 we have that $H^1_c(\mathfrak{X}, \mathbf{Q}_\ell^{\mathrm{ac}})$ surjects onto $\mathrm{Ind}_{\mathcal{K}_{x,n}}^{\mathrm{GL}_2(F) \times B^\times} H^1(\overline{\mathfrak{Z}}^{\mathrm{cl}}, \mathbf{Q}_\ell^{\mathrm{ac}})$. Thus the genus of $\overline{Z}_{x,n}^{\mathrm{cl}}$ is $(q-1)/2$, which forces the equation of $\overline{Z}_{x,n}$ to be $Y^q - Y = X^2$.

Case (4): x is ramified and n is even. The argument of case (ii) can be mimicked to show that the image of $\mathfrak{Z}_{x,n}$ in $\mathfrak{X}(K'_{x,n})$ is a disc. This time, if $v_E(\alpha) = -n$ then $(\mathfrak{A}_x, n, \alpha)$ is never a simple stratum, so there cannot be any ψ_α appearing in $H^1(\overline{Z}_{x,n}^{\mathrm{cl}}, \overline{\mathbf{Q}}_\ell^{\mathrm{ac}})$: it follows that $\overline{Z}_{x,n}^{\mathrm{cl}}$ is a disc. $\square$

6. A semistable covering of the Lubin-Tate tower

6.1 A dendritic filtration of $\mathrm{GL}_2(F)$

We use canonical lifts to define a dendritic filtration of the group $\mathrm{GL}_2(F)$, as in Defn. 2.9. This requires a $\mathrm{GL}_2(F)$-equivariant cycle-free graph Γ. For the vertex set of Γ, we take equivalence classes of pairs (x, n), where $x \in \mathfrak{X}(\pi^\infty)^{\mathrm{can}}$ and $n \geqslant 0$. Two pairs $v = (x, n)$ and $w = (y, n)$ shall be *equivalent* if one of the following conditions holds:

(i) $n = 0$, and $\mathfrak{A}_x = \mathfrak{A}_y$.

(ii) $n > 0$, and there exists $g \in \mathcal{L}_{x,n}^\times$ for which $x = y^g$.

(This relation is indeed symmetric.) An edge in Γ points from $v = (x, n)$ to $w = (y, m)$ under one of the following circumstances:

(i) $m = n = 0$, x is unramified, y is ramified, and $\mathfrak{A}_y \subset \mathfrak{A}_x$.

(ii) (y, m) is equivalent to $(x, n+1)$.

Then Γ admits an action of $\mathrm{GL}_2(F) \times B^\times$, wherein the stabilizer of $v = (x, n)$ is $\mathcal{K}_{x,n}$.

If $v = (x, n)$, we define $K_v = K_{x,n+1}$ and $\mathfrak{Z}_v = \mathfrak{Z}_{x,n}$.

6.2 The $\mathfrak{Z}_v$ account for all the cohomology

Let

$$\tilde{H}_{\mathrm{aff}}^1 = \bigoplus_v \mathrm{Ind}_{K_v \cap \mathrm{SL}_2(F)}^{K_v} \mathrm{Res}_{K_v \cap \mathrm{SL}_2(F)}^{K_v} H^1(\overline{\mathfrak{Z}}_{x,n}^{\mathrm{cl}}, \mathbf{Q}_\ell^{\mathrm{ac}}).$$

The sum ranges over vertices v of the graph Γ. Then $\tilde{H}_{\mathrm{aff}}^1$ admits an action of the subgroup of $\mathrm{GL}_2(F) \times B^\times$ consisting of those pairs (g, b) with $|\det g| = |N_{B/F}(b)|$. Write H_{aff}^1 for the induction of $\tilde{H}_{\mathrm{aff}}^1$ up to $\mathrm{GL}_2(F) \times B^\times$.

Suppose Π (resp., Π') is an admissible irreducible representation of $\mathrm{GL}_2(F)$ (resp., a smooth irreducible representation of $B^\times$). By Thm. 4.6, the following are equivalent:

(i) $\Pi \otimes \Pi'$ is contained in H_{aff}^1.

(ii) Π is supercuspidal and $\Pi' \cong \mathrm{JL}(\check{\Pi})$.

Let H_c^1 be the compactly supported cohomology of the Lubin-Tate tower of curves, as in §3.7. Let H_{cusp}^1 be the maximal supercuspidal quotient of H_c^1. By Thm. 3.3 we have the following

PROPOSITION 6.1. *There exists a surjection of $\mathrm{GL}_2(F) \times B^\times$-modules $H_{aff}^1 \to H_{cusp}^1$.*

We will use Prop. 6.1 to show:

PROPOSITION 6.2. *Suppose $K \subset K_v$ is a compact open subgroup. There exists a Zariski subaffinoid $\mathfrak{Z}_v^\circ \subset \mathfrak{Z}_v$ whose inverse image in $\mathfrak{X}(K)$ is a disjoint union of connected components $\coprod_i \mathfrak{Z}_{v,i}$, each of which has nonsingular reduction. Furthermore, the morphism $\overline{\mathfrak{Z}}_{v,i}^{cl} \to \overline{\mathfrak{Z}}_v^{cl}$ is purely inseparable for each i.*

Proof. Suppose $v = (x, n)$. It suffices to prove the Proposition for all K of the form $K = U^m = U_{\mathfrak{A}_x}^m$, where m is sufficiently large. Note that U^m is normal in K_v, so that $\mathfrak{X}(U^m)$ admits an action of K_v.

Let $\phi_m \colon \mathfrak{X}(U^m) \to \mathfrak{X}(K_v)$ be the quotient morphism. Using the main result of [Col03], produce a K_v-equivariant semistable covering $\{W_j\}_{j \in J}$ of $\mathfrak{X}(U^m)$. Then $\{\phi_m(W_j)\}_{j \in K_v \backslash J}$ is a semistable

covering of $\mathfrak{X}(K_v)$. By Lemma 5.1 there exists $j \in J$ for which a Zariski subaffinoid $\mathfrak{Z}_v^\circ \subset \mathfrak{Z}_v$ is contained in an underlying affinoid of $\phi(W_j)$. The preimage $\phi_m^{-1}(\mathfrak{Z}_v^\circ) \subset \mathfrak{X}(U^m)$ is a disjoint union of underlying affinoids of W_i, where i runs through the coset $K_v j \subset J$. We will show that the reduction of the quotient morphism $\phi_m^{-1}(\mathfrak{Z}_v^\circ) \to \mathfrak{Z}_v^\circ$ is purely inseparable when restricted to each connected component.

Choose a connected component $\mathfrak{Z}_v(m) \subset \phi_m^{-1}(\mathfrak{Z}_v^\circ)$, and let H_m be the stabilizer of $\mathfrak{Z}_v(m)$ in K_v. The connected components of $\mathfrak{X}(U^m)$ are a principal homogeneous space for the group $\mathcal{O}_F^\times / \det U^m$; this implies that $\det H_m \subset \det U^m$. The quotient of $\mathfrak{Z}_v(m)$ by H_m is $\mathfrak{Z}_v$, so we have an injection of $\mathbf{Q}_\ell[H_m]$-modules $\operatorname{Res}_{H_m}^{K_v} H^1(\overline{\mathfrak{Z}}_v^{\mathrm{cl}}, \mathbf{Q}_\ell) \hookrightarrow H^1(\overline{\mathfrak{Z}}_v(m)^{\mathrm{cl}}, \mathbf{Q}_\ell)$, which in turn induces an injection of $\mathbf{Q}_\ell[K_v]$-modules

$$\operatorname{Ind}_{H_m}^{K_v} \operatorname{Res}_{H_m}^{K_v} H^1(\overline{\mathfrak{Z}}_v^{\mathrm{cl}}, \mathbf{Q}_\ell) \hookrightarrow H^1(\overline{\phi_m^{-1}(\mathfrak{Z}_v^\circ)}^{\mathrm{cl}}, \mathbf{Q}_\ell) \tag{6.2.1}$$

On the other hand since $\det H_m \subset U^m$ we have an injection of $\mathbf{Q}_\ell[K_v]$-modules

$$\operatorname{Ind}_{K_v \cap \det^{-1}(\det U^m)}^{K_v} \operatorname{Res}_{\det^{-1}(\det U^m)}^{K_v} H^1(\overline{\mathfrak{Z}}_v^{\mathrm{cl}}, \mathbf{Q}_\ell) \hookrightarrow \operatorname{Ind}_{H_m}^{K_v} \operatorname{Res}_{H_m}^{K_v} H^1(\overline{\mathfrak{Z}}_v^{\mathrm{cl}}, \mathbf{Q}_\ell) \tag{6.2.2}$$

Since $\phi^{-1}(\mathfrak{Z}_v^\circ)$ is a disjoint union of underlying affinoids of wide opens appearing in a semistable covering of $\mathfrak{X}(U^m)$, Prop. 2.7 shows we have a surjection of $\mathbf{Q}_\ell[K]$-modules

$$H_c^1(\mathfrak{X}(U^m), \mathbf{Q}_\ell) \to H^1(\overline{\phi^{-1}(\mathfrak{Z}_v^\circ)}, \mathbf{Q}_\ell)$$

We repeat this procedure for larger m, and choose the connected components $\mathfrak{Z}_v(m)$ in a compatible manner, so that $H_{m+1} \subset H_m$; let $H = \bigcap_m H_m \subset K_v \cap \mathrm{SL}_2(F)$. Take direct limits along m, and then direct sums over vertices v, to arrive at a $\mathrm{GL}_2(F) \times B^\times$-equivariant injection

$$H_{\mathrm{aff}}^1 \hookrightarrow \bigoplus_v \varinjlim_m H^1(\overline{\phi_m^{-1}(\mathfrak{Z}_v^\circ)}^{\mathrm{cl}}, \mathbf{Q}_\ell)$$

and a $\mathrm{GL}_2(F) \times B^\times$-equivariant surjection

$$H_{\mathrm{cusp}}^1 \to \bigoplus_v \varinjlim_m H^1(\overline{\phi_m^{-1}(\mathfrak{Z}_v^\circ)}^{\mathrm{cl}}, \mathbf{Q}_\ell).$$

Comparing with Prop. 6.1 shows that both maps are isomorphisms, as are the maps in Eqs. (6.2.1) and (6.2.2). It follows that for each m, $H_m = K_v \cap \det^{-1}(\det U^m)$. We also find that the quotient morphism $\overline{\mathfrak{Z}}_v(m) \to \overline{\mathfrak{Z}}_v^\circ$ induces an isomorphism $H^1(\overline{\mathfrak{Z}}_v^{\mathrm{cl}}, \mathbf{Q}_\ell) \to H^1(\overline{\mathfrak{Z}}_v(m)^{\mathrm{cl}}, \mathbf{Q}_\ell)$, which shows that $\overline{\mathfrak{Z}}_v(m)^{\mathrm{cl}} \to (\overline{\mathfrak{Z}}_v^\circ)^{\mathrm{cl}}$ is purely inseparable as required. $\qquad \square$

6.3 Definition of the wide opens

For each vertex v of Γ we define a wide open $W_v \subset \mathfrak{X}(K_v)$ as follows.

If $v = (x, 0)$ and x is unramified: without loss of generality suppose that $\mathfrak{A}_x = M_2(\mathcal{O}_F)$, so that $\mathfrak{X}(K_v) = \mathfrak{X}(\pi)$. Let

$$W_v = \left\{ |u(z)| < |\pi|^{1/2} \right\} \setminus \bigcup_y \mathrm{red}^{-1}(y),$$

where y runs over unramified canonical points lying in $\mathfrak{Z}_{x,0}$ and $\mathrm{red} : \mathfrak{Z}_{x,0} \to \overline{\mathfrak{Z}}_{x,0}$ is the reduction map.

If $v = (x, 0)$ and x is ramified: without loss of generality suppose that $\mathfrak{A}_y = \begin{pmatrix} \mathcal{O}_F & \mathcal{O}_F \\ \mathfrak{p}_F & \mathcal{O}_F \end{pmatrix}$. Let

$$W_v = \left\{ |\pi|^{1/(q+1)} < |u(z)| < |\pi|^{q/(q+1)} \right\} \setminus \bigcup_y \mathrm{red}^{-1}(y),$$

where y runs over ramified canonical points lying in $\mathfrak{Z}_{x,0}$ and $\mathrm{red} : \mathfrak{Z}_{x,0} \to \overline{\mathfrak{Z}}_{x,0}$ is the reduction map. (These all have $|u(y)| = |\pi|^{1/2}$.)

If $v = (x, n)$ with $n \geqslant 1$: Let $\mathrm{red}_{n-1} : \mathfrak{Z}_{x,n-1} \to \overline{\mathfrak{Z}}_{x,n-1}$ be the reduction map, and similarly for red_{n+1}. Let

$$W_v = \phi^{-1}\left(\mathrm{red}_{n-1}^{-1}(x) \right) \setminus \bigcup_y \mathrm{red}_n^{-1}(y),$$

where $\phi : \mathfrak{X}(K_{x,n+1}) \to \mathfrak{X}(K_{x,n})$ is the projection, and y runs over canonical points in $\mathfrak{Z}_{x,n}$.

Now suppose $e : v \to w$ is an edge of Γ. We define wide opens $W_{v,e}$ as follows:

If $v = (x, 0)$ and $w = (y, 0)$, where x is ramified and y is unramified and $\mathfrak{A}_x \subset \mathfrak{A}_y = M_2(\mathcal{O}_F)$: Let

$$U_{w,e} = \left\{ |\pi|^{1/2} < |u(z)| < |\pi|^{q/(q+1)} \right\} \subset \mathfrak{X}(K_w)$$

and let $U_{v,e}$ be the image of $U_{w,e}$ under $\mathfrak{X}(K_w) \to \mathfrak{X}(K_v)$.

If $v = (x, n)$ and $w = (x, n+1)$, let $\phi = \phi_e : \mathfrak{X}(K_w) \to \mathfrak{X}(K_v)$ be the projection, let

$$U_{w,e} = \phi^{-1}\left(\mathrm{red}_n^{-1}(x) \right) \setminus \mathfrak{Z}_w,$$

and let $U_{v,e} = \phi(U_{w,e})$.

By construction we have

$$\mathfrak{Z}_v = W_v \setminus \bigcup_e U_{v,e},$$

where e runs over all edges incident to v.

LEMMA 6.3. *Condition (SC4) holds for the family $\{W_v\}$ and $\{U_{v,e}\}$ as a covering for $\mathfrak{X}(\pi^\infty) \setminus \mathfrak{X}(\pi^\infty)^{\mathrm{can}}$. That is, for any $y \in \mathfrak{X}(\pi^\infty)$ which is not a canonical lift, exactly one of the following holds:*

(i) *There exists a unique vertex v such that the image of y in $\mathfrak{X}(K_v)$ lies in $\mathfrak{Z}_v$.*

(ii) *There exists a unique edge $e : v \to w$ such that the image of y in $\mathfrak{X}(K_v)$ lies in $U_{v,e}$.*

Proof. The region $\left\{ |u(z)| \leqslant |\pi|^{1/2} \right\}$ is a "fundamental domain" in $\mathfrak{X}(\pi^\infty)$, see [Far08]: after replacing the point y with a $\mathrm{GL}_2(F)$-translate, we may assume that $|u(y)| \leqslant |\pi|^{1/2}$, and it is clear that the $\mathfrak{Z}_v$ and $U_{v,e}$ cover this region, where v ranges over those vertices (x, n) with $\mathfrak{A}_x \subset M_2(\mathcal{O}_F)$. The uniqueness statement is also clear from the construction. $\square$

LEMMA 6.4. *For each pair of vertices $v \preceq w$, and every edge e incident to w, the inverse image $\phi_{v,w}^{-1}(U_{v,e})$ is a disjoint union of open annuli.*

Proof. Since the affinoids $\mathfrak{Z}_{x,n}$ already account for all of the cuspidal part of $H_c^1(\mathfrak{X}(\pi^\infty), \mathbf{Q}_\ell)$, the connected component of each $U_{v,e}$ must be isomorphic to $\mathbf{P}^1$ minus a disjoint union of closed discs, one for each end. By construction, each connected component of $U_{v,e}$ has two ends. A similar argument applies to $\phi_{v,w}^{-1}(U_{v,e})$. $\square$

Prop. 6.2 and Lemmas 6.3 and 6.4 together imply:

THEOREM 6.5. *The data $\{W_v\}$, $\{W_{v,e}\}$ constitute a coherent semistable covering of $\mathfrak{X}(\pi^\infty) \setminus \mathfrak{X}(\pi^\infty)^{\mathrm{can}}$ with respect to Γ.*

6.4 The stable reduction of the tower of Lubin-Tate curves: Figures

In Figures 1-3, we draw the graph Γ corresponding to the dentric filtration of $\mathrm{GL}_2(F)$ constructed in §6.1. Each vertex v is labeled with a nonsingular projective curve Z_v/k^{ac} appearing on the list of four curves in Thm. 1.1. The graph Γ admits an action of the triple product group $\mathrm{GL}_2(F) \times B^\times W_F$. If v is any vertex, let $H_v \subset \mathrm{GL}_2(F) \times B^\times \times W_F$ be its stabilizer; then H_v acts on Z_v. Write Z for the disjoint union of all the Z_v. Then Z realizes the local Langlands correspondence in the following sense: whenever Π is an irreducible admissible representation of $\mathrm{GL}_2(F)$, and Π' and σ are irreducible representations of $B^\times$ and W_F, respectively, then

$$\mathrm{Hom}\left(\Pi \otimes \Pi' \otimes \sigma, H^1\left(Z, \mathbf{Q}_\ell^{\mathrm{ac}}\right)\right) \neq 0$$

if and only if Π is supercuspidal, $\Pi' = \mathrm{JL}(\check{\Pi})$, and $\sigma = \mathcal{H}(\pi)$.

We sketch a procedure for calculating the dual graph corresponding to the special fiber of a stable model of the classical modular curve $X_n = X(\Gamma(p^n) \cap \Gamma_1(N))$. First one must calculate the quotient $K_n \backslash \Gamma$, where $K_n = 1 + p^n M_2(\mathbf{Z}_p)$ is the principal congruence subgroup. The image of a vertex v in the quotient is labeled with the nonsingular projective curve constructed by quotienting Z_v by $H_v \cap K_n$. For almost every v, the quotient is rational. The quotient graph $K_n \backslash \Gamma$ has finitely many ends, and each end is (once one goes far enough) a ray consisting only of rational components. Erase all rational components lying on an end which corresponds to a canonical lift. The remaining ends correspond to the boundary of $\mathfrak{X}(p^n)$; these are in bijection with $\mathbf{P}^1(\mathbf{Z}/p^n\mathbf{Z})$. For each $b \in \mathbf{P}^1(\mathbf{Z}/p^n\mathbf{Z})$, erase all rational components lying on the end corresponding to b, and let v_b be the unique non-rational vertex which is adjacent to one of the vertices just erased. Call the resulting graph Γ_n.

Let $\mathrm{Ig}(p^n)$ denote the nonsingular projective model of the Igusa curve parameterizing elliptic curves over $\mathbf{F}_p^{\mathrm{ac}}$ together with Igusa p^n structures together with a point of order N. Draw $\mathbf{P}^1(\mathbf{Z}/p^n\mathbf{Z})$ many vertices w_b, and label each with $\mathrm{Ig}(p^n)$. For each $b \in \mathbf{P}^1(\mathbf{Z}/p^n\mathbf{Z})$, attach $\#X_1(N)(\mathbf{F}_p)^{\mathrm{ss}}$ copies of Γ_n to w_b in such a way that the vertex w_b is adjacent to each v_b. Finally, blow down any superfluous rational components. The result is a finite graph representing the special fiber of a stable model of X_n.

References

BH05a Colin J. Bushnell and Guy Henniart, *The essentially tame local Langlands correspondence. I*, J. Amer. Math. Soc. **18** (2005), no. 3, 685–710 (electronic).

BH05b ———, *The essentially tame local Langlands correspondence. II. Totally ramified representations*, Compos. Math. **141** (2005), no. 4, 979–1011.

BH06 ———, *The local Langlands conjecture for* $\mathrm{GL}(2)$, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006.

BK93 Colin J. Bushnell and Philip C. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993.

BL85 Siegfried Bosch and Werner Lütkebohmert, *Stable reduction and uniformization of abelian varieties. I*, Math. Ann. **270** (1985), no. 3, 349–379.

Boy99 P. Boyer, *Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale*, Invent. Math. **138** (1999), no. 3, 573–629.

Bro95 Paul Broussous, *Le dual admissible d'une algèbre à division locale: une paramétrisation par des types simples à la Bushnell et Kutzko*, Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], 1995/18, Univer-
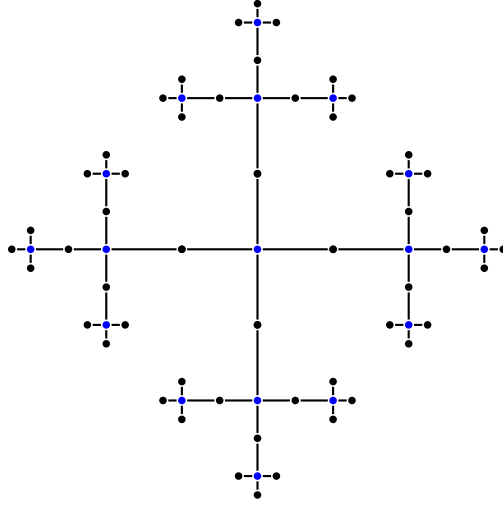
FIGURE 1. The "depth zero" subgraph of Γ, consisting of the vertices $v = (x, 0)$. The blue vertices are unramified. Each represents a copy of the nonsingular projective curve with affine model $xy^q - x^q y = 1$. The stabilizer of any particular blue vertex in $\mathrm{GL}_2(F)$ is conjugate to $\mathrm{SL}_2(\mathcal{O}_F)$. The black vertices are ramified. Each represents a rational component. The stabilizer of any black vertex in $\mathrm{SL}_2(F)$ is conjugate to an Iwahori subgroup.
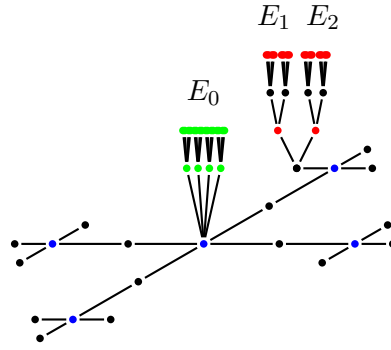

FIGURE 2. Here the depth zero subgraph of Γ is shown with several wild vertices to reveal structure. The green vertices are unramified; each represents a copy of the nonsingular (disconnected) projective curve with affine model $y^q + y = x^{q+1}$. The red vertices are ramified; each represents a copy of the nonsingular projective curve with affine model $y^q - y = x^2$. The wild vertices (x, n) labeled with an E_i are those for which x belongs to $\mathfrak{X}^{E_i}$. Here E_0, E_1, E_2 are the three quadratic extensions of F, with E_0/F unramified.
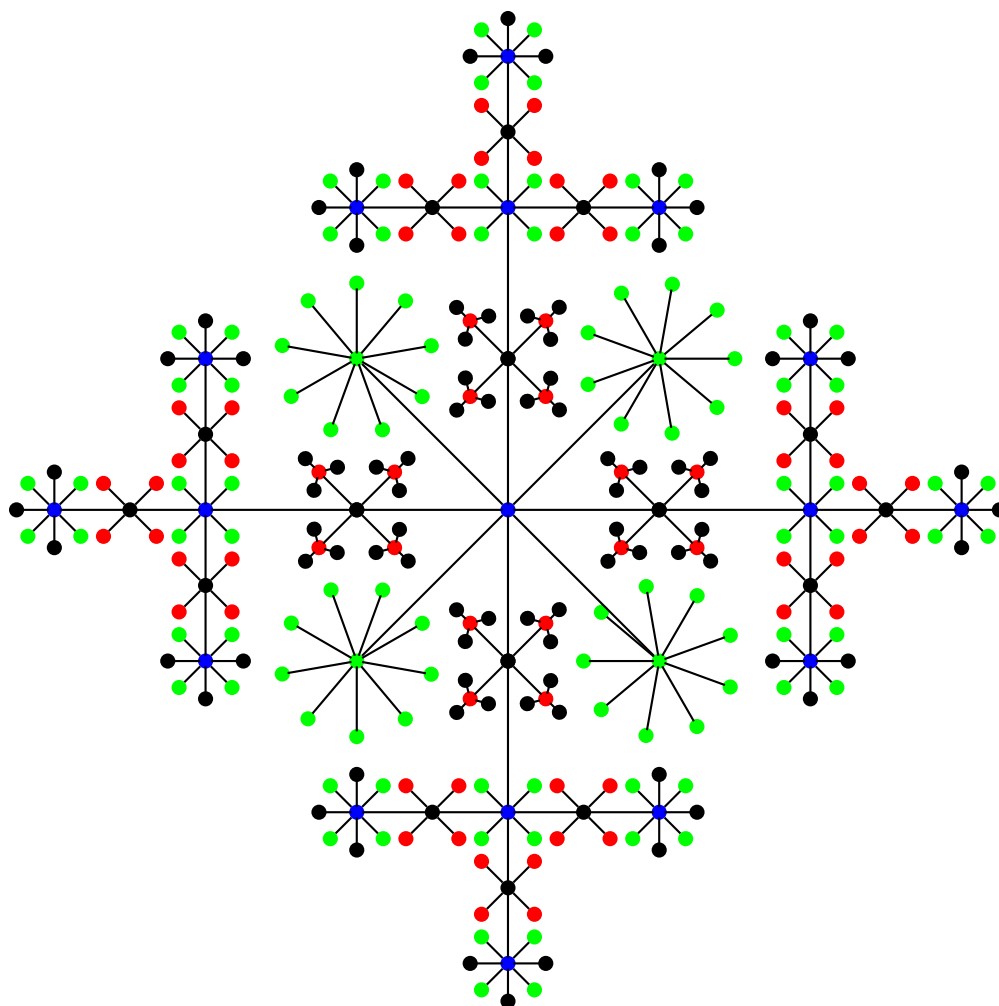
FIGURE 3. Dual graph of the reduction of the tower of Lubin-Tate curves: complete picture.

sité Louis Pasteur Département de Mathématique Institut de Recherche Mathématique Avancée, Strasbourg, 1995, Thèse, Université de Strasbourg I (Louis Pasteur), Strasbourg, 1995.

BW04   Irene I. Bouw and Stefan Wewers, *Stable reduction of modular curves*, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 1–22.

Car83   Henri Carayol, *Sur les représentations ℓ-adiques attachees aux formes modulaires de Hilbert*, C. R. Acad. Sci. Paris. **296** (1983), no. 15, 629–632.

Car86   H. Carayol, *Sur les représentations ℓ-adiques associées aux formes modulaires de Hilbert*, Annales scientifiques de l'É.N.S. **19** (1986), no. 3, 409–468.

CM10   Robert F. Coleman and Ken McMurdy, *Stable reduction of $X_0(Np^3)$*, Algebra and Number Theory **4** (2010), no. 4, 357–431, With an appendix by Everett W. Howe.

Col03   Robert F. Coleman, *Stable maps of curves*, Doc. Math. (2003), no. Extra Vol., 217–225 (electronic), Kazuya Kato's fiftieth birthday.

DR73   P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Lecture Notes in Mathematics **349** (1973), 143–316.

Dri74   V. G. Drinfel′d, *Elliptic modules*, Mat. Sb. (N.S.) **94(136)** (1974), 594–627.

Edi90   Bas Edixhoven, *Minimal resolution and stable reduction of $X_0(N)$*, Ann. Inst. Fourier (Grenoble) **40** (1990), no. 1, 31–67.

Far08   Laurent Fargues, *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques*, L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld, Progr. Math., vol. 262, Birkhäuser, Basel, 2008, pp. 1–325.

Gro86   Benedict H. Gross, *On canonical and quasicanonical liftings*, Invent. Math. **84** (1986), no. 2, 321–326.

Har02   M. Harris, *On the local langlands correspondence*, Proceedings of the ICM, Beijing **2** (2002), 583–598.

Haz78   Michiel Hazewinkel, *Formal groups and applications*, Pure and Applied Mathematics, vol. 78, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

Hen93   Guy Henniart, *Correspondance de Jacquet-Langlands explicite. I. Le cas modéré de degré premier*, Séminaire de Théorie des Nombres, Paris, 1990–91, Progr. Math., vol. 108, Birkhäuser Boston, Boston, MA, 1993, pp. 85–114.

HT01   Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

Ill94   Luc Illusie, *Autour du théorème de monodromie locale*, Astérisque (1994), no. 223, 9–57, Périodes p-adiques (Bures-sur-Yvette, 1988).

LT65   Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387.

Str08a   Matthias Strauch, *Deformation spaces of one-dimensional formal modules and their cohomology*, Adv. Math. **217** (2008), no. 3, 889–951.

Str08b   ———, *Geometrically connected components of Lubin-Tate deformation spaces with level structures*, Pure Appl. Math. Q. **4** (2008), no. 4, part 1, 1215–1232.

Weia   Jared Weinstein, *Explicit non-abelian Lubin-Tate theory for GL(2)*, preprint (2009), available at http://arxiv.org/abs/0910.1132.

Weib   ———, *Good reduction of affinoids on the Lubin-Tate tower*, preprint (2009), available at http://arxiv.org/abs/1001.3226.

Wei10   ———, *The local Jacquet-Langlands correspondence via Fourier analysis*, J. Théor. Nombres Bordeaux **22** (2010), no. 2, 483–512.

Yos10   Teruyoshi Yoshida, *On non-abelian Lubin-Tate theory via vanishing cycles*, Advanced Studies in Pure Mathematics (2010), no. 58, 361–402.

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